MARIA COLLEGE OF ENGINEERING AND TECHNOLOGY, ATTOOR.

B.E.Degree Examination

DISCREAT MATHEMATICS

UNIT IV

ALGEBRAIC STRUCTURES

2 MARK QUESTIONS

1.Let $E = \{2, 4, 6, \ldots\}$.Show that $\{E, +\}$ & $\{E, \times\}$ are semigroups but not monoids?

Let $E = \{2,4,6, \ldots\}$ be the set of all even numbers

Then $\{E, +\}$ is a semigroups

But $\{E, +\}$ is not a monoid since the additive identity element $0 \notin E$

Also $\{E, \times\}$ is a semi groups

But $\{E, \times\}$ is not a monoid since the multiplicative identity element $1 \notin E$

2.Give an example of semi groups which is not a monoid ?

$(N, +)$ is a semigroups ,where $N = \{1,2,3,4,5,6, \ldots\}$.It will not to be a monoid since it does not contain the additive identity element 0.

3. Define cyclic monoid

A monoid $(m,*)$ is said to be cyclic ,if every element of m is of the form a^n, $a \in m$, and n is a integer .

$(ie)\ x = a^n$

Such a cyclic monoid $(m,*)$ is said tp be generated by the element a .Here a is called the generater of the cyclic monoid.

4. Define semi group homomorphism

Let $(S,*)$ and (T, Δ) be any two semigroup with binary operation $*$ and Δ respectively

A mapping

$f: S \longrightarrow T$

Is said to be a semi group Homomorphism ,if

$f(a * b) = f(a) \Delta f(b)$ for all $a, b \in S$

5.Give an example of semi group Homomorphism?

Let $(N, +)$ and $(z_m, +_m)$ be any two semi group S.

Let us define a map
$f: N \longrightarrow z_m$

By $f(a) = [a]_m$, for all $a \in N$

Then $f(a + b) = [a + b]_m$

$= [a] +_m [b]$

$= f(a) +_m f(b)$

$\therefore f$ is a semigroup homomorphism.

6. Show that composition semi group Homomorphism is also a semi group Homomorphism?

Let $a, b \in S$.Then

$(h\ o\ g)(a * b) = h\big(g(a * b)\big)$

$= h(g(a) \Delta g(b))$ $\quad (\because g \text{ is a homomorphism})$

$= h(g(a)) \oplus h\big(g(b)\big)$ $\quad (\because h \text{ is a homomorphism})$

$= (h\ o\ g)(a) \oplus (h\ o\ g)(b)$

Hence $h\ o\ g$ is a semi group homomorphism from $(S, *)$ to $(V, \oplus)$.

7.Let Let $(s, *)$ and (T, Δ) be two semigroup and $f: S \longrightarrow T$ be a semigroup homomorphism .If $a \in S$ is idempotent ,then show that $g(a)$ is alsoi idempotent?

Given a is idempotent

$\therefore a * a = a \ldots \ldots (1)$

Hence $[g(a)]^2 = g(a) \Delta g(b)$

$= g(a * a)$

$= g(a)$ $\quad by(1)$

Since$[g(a)]^2 = g(a)$, $g(a)$ is idempodent

8.Let $S = Q \times Q$ be the set of all ordered pairs of rational numbers and given by

$(\boldsymbol{a}, \boldsymbol{b}) * (\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{ax}, \boldsymbol{ay} + \boldsymbol{b})$.**Find the identity element of $\boldsymbol{S}$?**

Let (e_1, e_2) be the identity element of $(S,*)$

Then for any $(a, b) \in S$,

$(a, b) * (e_1, e_2) = (a, b)$

$$= (ae_1 \, ae_2 + b) = (a, b)$$

$\Longrightarrow ae_1 = a$ and $ae_2 + b = b$

$e_1 = 1$ and $e_2 = \frac{b-b}{a} = 0, (a \neq 0)$

∴the identity element= (e_1, e_2)=$(1{,}0)$

9. Show that the identity element of a group is unique?

Let $(G,*)$ be a group.

Let e_1 and e_2 be two identity element in G .

Suppose e_1 is the identity then $e_1 * e_2 = e_2 * e_1 = e_2 \dots \dots (1)$

Suppose e_2 is the identity then $e_2 * e_1 = e_1 * e_2 = e_1 \dots \dots (2)$

From (1) and (2) we get $e_1 = e_2$

∴the identity element is unique.

10. Show that the inverse element of a group is unique ?

Let $(G,*)$ be a group.

Let $a \in G$ and e be the identity of G

Let ${a_1}^{-1}$ and ${a_2}^{-1}$ be the two different inverse of the same element.

$\therefore \quad {a_1}^{-1} * a_1 = a_1 * {a_1}^{-1} = e$

${a_2}^{-1} * a_2 = a_2 * {a_2}^{-1} = e$

$({a_1}^{-1} * a_1) * {a_2}^{-1} = e * {a_2}^{-1} = {a_2}^{-1} \dots \dots (1)$

${a_1}^{-1} * (a_2 * {a_2}^{-1}) = {a_1}^{-1} * e = {a_1}^{-1}$........(2)

From (1) and (2),we get

${a_1}^{-1} = {a_2}^{-1}$

∴there will be no two different inverses for the same element.

11.Define isomorphism?

A mapping f from a group $(G,*)$ to a group (G', Δ) is said to be an isomorphism if

(i) f is homomorphism

$(ie)\ f(a*b) = f(a)\Delta f(b)$ for all $a, b \in G$

(ii) f is one-one(injective)
(iii) f is onto surjective
in other words a bijective homomorphism is said to be an isomorphism.

12.Define normal subgroup?

Let H be a subgroup of G under $*$.Then H is said to be G, for every $x \in G$ and for $h \in H$,

If $x * h * x^{-1} \in H$

$x * H * x^{-1} \in H$

Alternatively, a subgroup H of G is called a normal subgroup of G if $x * h = h * x$ for all $x \in G$

13. Prove that in an abelian group $(ab)^2 = a^2b^2$?

$(ab)^2 = (ab)(ab)$

$= a(ba)b$

$= a(ab)b$

$= (aa)(bb)$

$=a^2b^2$

14.Show that in a group G,$x^2 = x$ if and only if $x = e$?

Clearly $e^2 = ee = e$

Conversely $x^2 = x$

$xx = xe$

$x = e$

15. Consider the group $Z_4 = \{[0], [1], [2], [3]\}$ of integers modulo 4.Let $H = \{[0], [2]\}$ be a subgroup of Z_4 under $+_4$(addition mod4).Find the left cosets of H.

Then the left cosets of H are

$[0]+H=\{[0],[2]\}=H$

$[1]+H=\{[1],[3]\}$

$[2]+H=\{[2],[4]\}=\{[2],[0]\}=\{[0],[2]\}=H$

$[3]+H=\{[3],[5]\}=\{[3],[1]\}=\{[1],[3]\}=\ [1]+H$

$[0]+H=\quad [2]+H=H$

And $[1]+H=[3]+H$ are the two distinct left cosets of H in Z_4

16.When an element in a ring is said to be a zero devisor?

If a and b are two non zero elements of a ring R such that $a.b=0$, then a and b are called zero divisors.

17.Define integral domain .Give an example?

A commutative ring $(R,+,.)$ with identity and without zero divisors is called an integral domain.

$(Z,+,.)$ is an integral domain.

17.Let $[\boldsymbol{R},+,.]$ be a ring ,For $\boldsymbol{a},\boldsymbol{b}\in\boldsymbol{R}$ show that $(\boldsymbol{a}+\boldsymbol{b})^2=\boldsymbol{a}^2+\boldsymbol{a}.\boldsymbol{b}+\boldsymbol{b}.\boldsymbol{a}+\boldsymbol{b}^2$where $\boldsymbol{a}^2=\boldsymbol{a}.\boldsymbol{a}$?

For $a,b\in R$,we have

$(a+b)^2=(a+b).(a+b)$

$=(a+b).a+(a+b).b$

$=a.a+a.b+b.a+b.b$

$=a^2+a.b+b.a+b^2$

18. Let $[\boldsymbol{R},+,.]$ be a ring ,For $\boldsymbol{a},\boldsymbol{b}\in\boldsymbol{R}$ show that $\boldsymbol{a}.(\boldsymbol{b}-\boldsymbol{c})=\boldsymbol{a}.\boldsymbol{b}-\boldsymbol{a}.\boldsymbol{c}$

Clearly $a.(b-c)=a.\big(b+-(-c)\big)$

$=a.b+a.(-c)$

$=a.b-a.c$

19.If a homomorphism of a group $\boldsymbol{G}$ into a group $\boldsymbol{G}'$ then prove that f preserves identities?

Let $a\in G$

Let f be a homomorphism from $(G,*)$ into $(G',*)$

Clearly $f(a)\ \in G'$

Now, $f(a) * e' = f(a) \quad [\because e' - identity\ in\ G']$

$$= f(a * e) \quad [\because e' - identity\ in\ G]$$

$$= f(a) * f(e) \quad [f - homomorphism]$$

20.Define Ring with example?

An algebraic system $(R, +, .)$ is called a ring if the binary operations $+$ and . satisfies the following conditions:

(i) $(a + b) + c = a + (b + c)$, $a, b, c \in R$

(ii) There exists an element $0 \in R$ called zero element such that $a + 0 = 0 + a = a$ for all $a \in R$

(iii) For all $a \in R$,$a + (-a) = (-a) + a = 0, -a$ is the negative of a

(iv) $a + b = b + a$, for all $a, b \in R$

(v) $(a.b).c = a.(b.c)$ for all $a, b, c \in R$

(vi) The operation . is distributive over $+$,(ie) for any $a, b, c \in R$

$$a.(b + c) = a.b + a.c$$
$$(b + c).a = b.a + c.a$$

In other words if R is an abelian group under addition with the properties (v) and (vi) then R is a ring.

Example:

$(z, +, .)$; $(Q, +, .)$; $(R, +, .)$; $(C, +, .)$ …

Unit III Graphs

1.Define Graph?

A graph $G = (V, E, \phi)$ consists of a non empty set $V = \{V_1, V_2, \dots\}$ called the set of nodes (points,Vertices) of the graph $E = \{e_1, e_2, \dots \dots\}$ is said to be the set of edges of the graph and ϕ is a mapping from the set of edges E to set of ordered or unordered pairs of elements of V

2.Name the types of graphs?

(i) directed graph

(ii) undirected graph

(iii)Mixed graph

(iv) Multi graph
(v) Pseudo graph

3.For the following degree sequences 4,4,4,3,2 find if there exist a graph or not ?

Sum of the degree of all vertices=4+4+4+3+2=17

Which is an odd number

∴ such a graph does not exist.

4) Define a simple graph ?

A graph which has neither self loop nor parallel edges is called a simple graph.

5) Define directed graph ?

In a graph G=(V,E) in edge which is associated with an ordered pair of VxV is called a directed edge of G

6) The sum of all degrees of all the vertices of an directed graph is

2e

7) In an undirected graph the number of odd degree vertices is

Even

8) The maximum number of degrees of edges in a simple graph with n vertices is

n(n-1)/2

9) How many edges are there in a graph with 10 vertices each of degree 6 ?

We know that total the number of degrees is $\sum d(v)= 2e$

2e=10x6

2e=60

e=30

10) Can a simple graph with 15 vertices each of degree 5?

We know that total the number of degrees is $\sum d(v)= 2e$

2e=15x5

2e=75

e=75/2

11) How many vertices do a regular graph of degree 4 with 10 edges?

We know that total the number of degrees is $\sum d(v)= 2e$

Let n be the number of vertices and e be the number of edges

Ie, 4n=2x10

Ie,n=5

∴ The number of vertices in a regular graph of degree 4 with 10 edges

12) Does there exist a simple graph with five vertices of the following degree?

We know that in any graph the number of odd degree vertices is even

Ie, 1+1+1+1+1=5, not an even.

∴ Such graph does not exist.

13) The degree of a simple complete graph of n vertices is

n-1

14) Can a single vertex be a subgraph?

Yes.

15) Can a subgraph of a subgraph be subgraph?

Yes.

16) The sum of all entries in a row of a adjacency matrix is ……………………………….

The degree of the corresponding vertex.

17) If A be the adjacency matrix of a graph then A^2 and A^3 are

Symmetric

18) If G_1 and G_2 are isomorphic then G_1 and G_2 have

a) Same number of vertices.

b) Same number of edges

19) The simple graph with n vertices must be complete if the number of edges is

(n-1)x(n-2)/2

20) The number of edges of a simple graph with n vertices and K components is

(n-k)x(n-k+1)/2

21) If the simple graph G has V vertices and e edges how many edges does G^c have

(V(v-1)/2)-e edges

22) If the simple graph G has 4 vertices and 5 edges then how many edges does G^c have?

We know that G^c has (v(v-1)/2)-e edges. Here V=4,e=5

∴ the number of edges is (4(4-1)/2)-5=6-5=1

Hence G^c has only one edge .