PROBABILITY AND QUEUEING THEORY

UNIT I

RANDOM VARIABLES

PART-A

1. **A random sample 'X' has the probability function**

X	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
P(x)	**a**	**3a**	**5a**	**7a**	**9a**	**11a**	**13a**	**15a**	**17a**

Determine the value of a

Soln:

Wkt if P(x) is the probability mass function then $\sum_{i=1}^{\infty} p(x_i) = 1$

i.e a+3a+5a+7a+9a+11a+13a+15a+17a=1

81a=1

a=$\frac{1}{81}$

2. **If X is a continuous random variable whose probability density function is given by** $f(x) = \begin{cases} c(4x - 2x^2), & 0 < x < 2 \\ 0, & otherwise \end{cases}$ **what is the value of c?**

Wkt $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_0^2 c(4x - 2x^2)dx = 1$$

2c$\int_0^2 (2x - x^2)dx = 1$

2c$\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^2 = 1$

$2c\left|4-\frac{8}{3}\right| = 1$

$2c\left[\frac{4}{3}\right]=1$ □ $c=\frac{3}{8}$

3. **Given that the p.d.f of a R.V X is $f(x) = Kx, 0 < x < 1$ find K and P(X>0.5)**

W.k.t $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_0^1 Kxdx = 1 \;□\; K\left[\frac{x^2}{2}\right]_0^1 = 1$$

K=2

Also P(X>0.5)= $\int_{0.5}^{\infty} f(x)dx = \int_{0.5}^{1} 2xdx$

$=2\left[\frac{x^2}{2}\right]_{0.5}^{1} = [1 - 0.25] = 0.75$

4. **In a continuous random variable X having the p.d.f $f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2 \\ 0, & otherwise \end{cases}$**

   Find P(0<X≤ 1)

P(0<X≤ 1) $= \int_0^1 f(x)dx = \int_0^1 \frac{x^2}{3}dx$

$$= \frac{1}{3}\left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}\left|\frac{1}{3} - 0\right| = \frac{1}{9}$$

5. **For the following c.d.f F(x)== $\begin{cases} 0, & x < 0 \\ x, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$ find (i)P(X>0.2) (ii)P(0.2<X≤ 0.5)**

P(X>0.2)=1-P(X≤0.5)

=1-F(0.2)=1-0.2=0.8

(ii)$P(0.2<X\leq 0.5)=F(0.5)-F(0.2)$

$=0.5-0.2=0.3$

6. **Find the m.g.f of the distribution is given by** $f(x) = \begin{cases} \lambda e^{-\lambda x}, x > 0 \\ 0, \quad otherwise \end{cases}$

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

$$= \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^{\infty} e^{-(\lambda - t)x} dx$$

$$= \lambda \left[\frac{e^{-(\lambda-t)x}}{-(\lambda-t)} \right]_0^{\infty} = -\frac{\lambda}{\lambda - t}[-e^{-\infty} - e^0] = -\frac{\lambda}{\lambda - t}(0 - 1)$$

$=\frac{\lambda}{\lambda - t}$

7. **Check whether the following data follow a binomial distribution or not. Mean=3; variance=4.**

Given Mean = np= 3 …………………………………….(1)

Variance=npq=4 ………………………………………..(2)

$$\frac{(2)}{(1)} \square \ \frac{npq}{np} = \frac{4}{3} > 1$$

Since q>1 which is not possible (0<q<1).The given data does not follow Binomial distribution.

8. **With the usual notation find p for a binomial random variate X if n=6 and if 9P(X=4)=P(X=2).**

For a binomial random variate 'X', P(X=x) = $nC_x p^x q^{n-x}$

Given 9P(X=4) = P(X=2).

$9x6C_4p^4q^2 = 6C_2p^2q^4$

$9p^2 = q^2 = (1-p)^2$

$9p^2 = 1 + p^2 - 2p$

$8p^2 + 2p - 1 = 0$

$P = \frac{1}{4}, -\frac{1}{2}$

But $p = -\frac{1}{2}$

a. $\therefore p = \frac{1}{4}$

9. If the mgf of a r.v. X is of the form $(0.4e^t + 0.6)^8$ What is the mgf of 3X+2.

The M.g.f of a Binomial distribution is $M_X(t) = (q + pe^t)^n$

Here q=0.6 ; p=0.4;n=8

Thus X follows a binomial distribution with p=0.4,n=8

Mean=np=8x0.4=3.2

$\therefore$ mgf of 3X+2 is given by $M_{3X+2}(t) = E\left(e^{(3X+2)t}\right) = e^{2t}E[e^{3Xt}]$

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$= e^{2t}(0.4e^{3t} + 0.6)^8$

10. Find the mgf of uniform distribution.

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

$= \int_a^b e^{tx} f(x)dx = \int_a^b e^{tx} \frac{1}{b-a} dx \quad \left|since \ \ f(x) = \frac{1}{b-a}, a < x < b\right|$

$= \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b = \frac{1}{(b-a)t}[e^{bt} - e^{at}]$

11. For a binomial distribution mean is 6 and S.D is $\sqrt{2}$.Find the first two terms of the distribution.

For a binomial distribution mean =np=6

Variance=npq

Given $S.D = \sqrt{npq} = \sqrt{2}$ □ $npq = 2$

$$=> 6q = 2 => q = \frac{1}{3}$$

$$p = 1 - q = 1 - \frac{1}{3} = \frac{2}{3}$$

$$np = 6 => n\frac{2}{3} = 6 => n = 9$$

The prob mass function of a binomial distribution is given by $P(X = r) = nC_r p^r q^{n-r}, r = 0,1,2, \ldots ; n$

Hence the first two terms are given by ,

$$nC_0 p^0 q^n, nC_1 pq^{n-1} \quad (i.e.,)\ 9C_0 \left(\frac{1}{3}\right)^9, 9C_1 \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^8$$

$$\left(\frac{1}{3}\right)^9, \quad 9\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^8$$

12. Define Binomial frequency distribution?

Let us suppose that n trials constitute an experiment. Then if this experiment is repeated N times , the frequency function of the binomial distribution is given by,

$$f(x) = Np(x) = N\ nC_x p^x q^{n-x}, x = 0,1,2, \ldots . n$$

The expected frequencies of 0,1,2, ….,n successes are given by the successive terms of $N(p+q)^n$

13. .Define Poisson distribution and state any two instances where Poisson distribution may be successfully employed.

A r.v X is said to follow poisson distribution if it assumes only non-negative values and its probability mass function is given by

$$P(X=x)=p(x)=\begin{cases}\frac{e^{-\lambda}\lambda^x}{x!}, x=0,1,2,3,\dots \\ 0, \quad otherwise\end{cases}$$ λ is known as the parameter of the poisson distribution.

<u>The instances where poisson distribution may be successfully employed</u>

1. Number of printing mistakes at each page of the book

2. Number of suicides reported in a particular day.

3. Number of defective items produced in a factory.

14. .**Define Geometric distribution**.

Suppose that independent trials, each having a probability p,0<p<1, of being a success , are performed until a success occurs. IF we let X equal number of trials required, then $P\{X=x\}=(q)^{x-1}p, x=1,2,3,\dots.$

15. .**Define Exponential distribution.**

A continuous random variable X is said to follow exponential distribution if its probability density function is given by $f(x)=\begin{cases}\alpha e^{-\alpha x}, x\geq 0, \alpha>0\\ 0 \quad , otherwise\end{cases}$

16. Define Gamma distribution

A continuous random variable X taking nonnegative values is said to follow gamma distribution , if its probability density function is given by

$$f(x) = \begin{cases} \frac{e^{-x}x^{\alpha-1}}{\Gamma(\alpha)}, \alpha > 0.0 < x < \infty \\ 0, otherwise \end{cases}$$ where α is the parameter of the distribution.

17. If 3% of the electric bulbs manufactured by a company are defective, find the probability that in a sample of 100 bulbs exactly 5 bulbs are defective ($e^{-3} = 0.0498$)

Let X be the r.v denoting the number of defective bulbs.

Given p(a bulb is defective)=$\frac{3}{100}$

P=0.03

n=100

λ=np=100x0.03=3

We know that P(X=x)= $\frac{e^{-\lambda}\lambda^x}{x!}$

P(exactly 5 bulbs are defective)=P(X=5)

$$=\frac{e^{-3}3^5}{5!}=\frac{0.0498\times243}{120} = 0.1008$$

18. If the probability is 0.05 that a certain kind of measuring device will show excessive drift , what is the probability that th esixth of these measuring devices tested will be the first to show excessive drift.

Here p=0.05 => q=1-0.05=0.95

X=6

W.k.t $P(X = x) = q^{x-1}p = (0.95)^5(0.05) = 0.0387$

19. Define Weibull distribution?

A random variable X is said to follow Weibull distribution with two parameters β>0,α>0 if the probability density function is given by $f(x) = \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}, x > 0$

20. The time in hours required to repair a machine is exponentially distributed with parameter λ=$\frac{1}{2}$ .What is the probability that a repair takes atleast 10 hrs given that its duration exceeds 9 hours.

Let X be the R.V. which represents the time to repair the machine.

 $P(X > 10/X > 9) = P(X > 1)$ [by memoryless property]

$$= e^{-\frac{1}{2}} = 0.6065$$

PART-B

1) A random variable X has the following probability distribution.

X	-2	-1	0	1	2	3
P(x)	0.1	K	0.2	2k	0.3	3k

Find i) the value of k

ii) Evaluate p(x<2) and p(-2<x<2)

iii) Find the cumulative distribution of X.

iv) Find the mean of X.

2) A probability curve y=f(x) has a range from 0 to ∞. If f(x) =e^{-x}, find the mean and variance and third moment about mean.

3) If the density function of a cumulative R.V. X is given by f(x) = $\begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ 3a - ax, & 2 \le 3 \le 3 \\ 0, & otherwise \end{cases}$.

i) Find the value of a.

ii) Find the value c.d.f of x.

4) For the triangular distribution f(x) = $\begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \\ 0, & otherwise \end{cases}$. Find the mean and variance.

5) The p.d.f. of the R.V. X follows the following probability law f(x) = $\frac{1}{2\theta} e^{\frac{-|x-\theta|}{\theta}}$, $-\infty < x < \infty$. Find the m.g.f of X . Hence find E(X) and Var(X).

6) If 10% of the screws produced by an automatic machine are defective, find the probability that out of 20 screws selected at random there are i) exactly 2 defectives. ii) atmost 3 defectives. iii)atleast 2 defectives. iv) between 1 and 3 defectives (inclusive).

7) Find the m.g.f, mean and variance of poisson distribution.

8) State and prove the memory less property of the exponential distribution.

UNIT II

Two Dimensional Random Variables

1. The bivariate random variable X and Y has the pdf

$\boldsymbol{f(x, y) = Kx^2(8 - y)}, \boldsymbol{x} < y < 2x, 0 < x < 2$. **Find K**

We know that if f(x,y) is a p.d.f, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dydx = 1$$

$$\Rightarrow \int_0^2 \int_x^{2x} Kx^2(8 - y)dydx = 1$$

$$\Rightarrow K\int_0^2 x^2 \left[8y - \frac{y^2}{2}\right]_x^{2x} dx = 1$$

⇨ $K\int_0^2 x^2\left(16x - \frac{4x^2}{2} - 8x + \frac{x^2}{2}\right)dx = 1$

⇨ $K\int_0^2 \left(16x^3 - 2x^4 - 8x^3 + \frac{x^4}{2}\right)dx = 1$

⇨ $K\int_0^2 \left(8x^3 - \frac{3}{2}x^4\right)dx = 1$

⇨ $K\left[\frac{8x^4}{4} - \frac{3x^5}{2.5}\right]_0^2 = 1$

⇨ $K\left|32 - \frac{48}{5}\right] = 1$

⇨ $K\frac{112}{5} = 1$

⇨ $K=\frac{5}{112}$

2. **The joint p.d.f of R.V. X and Y is given by $\boldsymbol{f(x,y) = kxye^{-(x^2+y^2)}, x} > 0, y > 0$.Find the value of k**

We know that if f(x,y) is a p.d.f, then

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dydx = 1$

 $\int_0^{\infty}\int_0^{\infty} kxye^{-(x^2+y^2)}dydx = 1$

⇨ $k\int_0^{\infty} ye^{-y^2}dy \int_0^{\infty} xe^{-x^2}dx = 1$

⇨ $k\frac{1}{2}\frac{1}{2} = 1 \; => k = 4$ $\left[\int_0^{\infty} xe^{-x^2}dx = \frac{1}{2}\right]$

3. **The joint pdf of (X,Y) is given by $\boldsymbol{f(x,y) = \frac{1}{4}, 0} < x,y < 2$ find P(X+Y≤ $\mathbf{1}$)**

P(X+Y≤ 1)=$\int_0^1\int_0^{1-y} f(x,y)dxdy$

$=\int_0^1\int_0^{1-y}\frac{1}{4}dxdy$

$=\frac{1}{4}\int_0^1 [x]_0^{1-y}dy$

$=\frac{1}{4}\int_0^1 (1-y)dy = \frac{1}{4}\left[y - \frac{y^2}{2}\right]_0^1$

$=\frac{1}{4}\left[1-\frac{1}{2}\right]=\frac{1}{8}$

4. **Give an example for conditional distribution**.

If joint pdf of (x,y)=f(x,y) then conditional pdf of X given Y is $f(x/y)=\frac{f(x,y)}{f(y)}$ *where* $f(y)$ *is the marginal pdf of Y*

5. **If X and Y have joint pdf** $f(x,y)=\begin{cases} x+y, & 0<x<1.0<y<1 \\ 0, & otherwise \end{cases}$ **check whether X and Y are independent.**

If X and Y are independent then f(x,y)=f(x)f(y)

The marginal density function of X is $f(x)=\int_{-\infty}^{\infty} f(x,y)dy$

$$=\int_0^1 (x+y)dy = \left|xy+\frac{y^2}{2}\right]_0^1 = x+\frac{1}{2}$$

The marginal density function of Y is $f(y)=\int_{-\infty}^{\infty} f(x,y)dx$

$$=\int_0^1 (x+y)dx = \left[\frac{x^2}{2}+xy\right]_0^1 = \frac{1}{2}+y$$

 $f(x)f(y)=\left(x+\frac{1}{2}\right)\left(\frac{1}{2}+y\right)$

$$= f(x,y)$$

Hence X and Y are not independent.

6. **If the joint pdf of (X,Y) is given by f(x,y)=2-x-y,0≤ $x \le y \le 1$,find E[XY].**

The marginal density function of X is is $f(x)=\int_{-\infty}^{\infty} f(x,y)dy$

$$=\int_x^1 (2-x-y)dy = \left|2y-xy-\frac{y^2}{2}\right]_x^1$$

$$=\frac{3}{2}(1-2x+x^2);0<x<1$$

E[X]= $\int_0^1 x\frac{3}{2}(1-2x+x^2)\,dx = \frac{3}{2}\int_0^1 (x-2x^2+x^3)dx$

$= \frac{3}{2}\left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_0^1 = \frac{1}{8}$

7. Determine the value of C such that the function f(x,y)=cxy and 0<x<3 and 0<y<3 satisfies the properties of a joint probability density function.

Since f(x,y) is a pdf, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dydx = 1$

$C\int_0^3\int_0^3 xydxdy = 1$

$C\int_0^3 y\left(\frac{x^2}{2}\right)_0^3 dy = 1 => c\int_0^3\left(\frac{9}{2}\right)ydy = 1$

$C\left(\frac{9}{2}\right)\left[\frac{y^2}{2}\right]_0^3 = 1 => \frac{81C}{4} = 1 => C = \frac{4}{81}$

8. Let X be a random variable withp.d.f f(x)=$\frac{1}{2}$, $-1 \le x \le 1$ and let $Y = X^2$.Find E[Y].

E[X]=$\int_{-\infty}^{\infty} xf(x)dx$

$=\int_{-1}^{1} x\frac{1}{2}dx = \frac{1}{2}\left(\frac{x^2}{2}\right)_{-1}^{1} = \frac{1}{2}\left(\frac{1}{2}-\frac{1}{2}\right) = 0$

E[Y]=E[X^2]=$\int_{-\infty}^{\infty} x^2 f(x)dx$

$=\int_{-1}^{1} x^2\frac{1}{2}dx = \frac{1}{2}\left(\frac{x^3}{3}\right)_{-1}^{1} = \frac{1}{2}\left[\frac{1}{3}+\frac{1}{3}\right]$

$=\frac{1}{2}\left[\frac{2}{3}\right] = \frac{1}{3}$

9. The j.d.f. of the random variables X and Y is given by $f(x,y) = \begin{cases} 8xy, & 0<x<1, 0<y<x \\ 0, & otherwise \end{cases}$ find$f_X(x)$

The marginal density function of X is $f_X(x) = f(x) = \int_{-\infty}^{\infty} f(x,y)dy$

$= \int_0^x 8xydy = 8x\left(\frac{y^2}{2}\right)_0^x = 4x(x^2 - 0)$

$= 4x^3, 0 < x < 1$

10. Find the acute angle between the two lines of regression.

Angle between the lines of regression is given by

$$tan\theta = \frac{1-r^2}{r}\left(\frac{\sigma_X\sigma_Y}{\sigma_X^2+\sigma_Y^2}\right)$$

11. The two equations of the variables X and Y are $x = 19.13 - 0.87y$ and $y = 11.64 - 0.50x$. Find the correlation coefficient between X and Y.

Given that the regression equation of X on Y is $x = 19.13 - 0.87y$

the regression coefficient of X on Y is $b_{xy} = -0.87$

The regression equation of Y on X is $y = 11.64 - 0.50x$.

the regression coefficient of Y on X is $b_{yx} = -0.50$

correlation coefficient $r = \pm\sqrt{b_{xy}b_{yx}} = \pm 0.6595$

12. Let (X,Y) be two dimensional random variable.Define covariance of (X,Y).If X and Y are independent ,what will be the covariance of (X,Y).

Cov(X,Y)=E[(X-E(X)(Y-E(Y)]

=E[XY]-E[X]E[Y]

If X and Y are independent , then cov(x,y)=0

13. Comment on the following: "The random variables X and Y are independent iff Cov(X,Y)=0"

If X and Y are independent , then cov(x,y)=0

=>X and Y are uncorrelated

If Cov(X,Y)=0 then X and Y are uncorrelated

=>X and Y need not be independent.

14. The regression equations of X on Y and Y on X are respectively 5x-y=22 and 64x-45y=24. Find the means of X and Y.

Since both the regression equations passes through $(\bar{x}, \bar{y})$ we get

$5\bar{x} - \bar{y} = 22$ ……………………………(1)

$64\bar{x} - 45\bar{y} = 24$ ……………………………(2)

(1)x45 => $225\bar{x} - 45\bar{y} = 990$ ……………………………(3)

(3)-(2) => $161\bar{x} = 966$ => $\bar{x} = \frac{966}{161} = 6$

The mean of X =6

Put $\bar{x}$=6 in (1)

(1) => 5(6)- $\bar{y}$ =22

$\bar{y} = 30 - 22 = 8$

The mean of Y=8

15. X and Y are independent random variables with variance 2 and 3. Find the variance of 3X+4Y.

Given that X and Y are independent r.v.s with variance 2 and 3

(i.e) var(X)=2 , var(Y)=3

Consider $\mathrm{Var}(3X + 4Y) = 3^2\mathrm{Var}(X) + 4^2\mathrm{Var}(Y)$

$= 9 \times 2 + 16 \times 3 = 66$

16. State the central limit theorem for independence and identically distributed random variables.

If $X_1, X_2, \ldots \ldots \ldots X_n$ be a sequence of independence and identically distributed random variables with $E[X_i] = \mu$ and $Var(X_i) = \sigma^2$, i=1,2,,..... and if $S_n = X_1 + X_2 + \cdots \ldots . . X_n$, then under certain general conditions, S_n follows a normal distribution with mean n μ and variance nσ^2 as n tends to infinity

17. Write the applications of central limit theorem.

1. Central limit theorem provides on simple method for computing approximate probabilities of sums of independent random variables
2. It also gives us the wonderful fact that the empirical frequencies of so many natural populations exhibit a bell shaped curve

18. *If Y=-2x+3, find the cov(x,y)*

Given y=-2x+3

Cov(X,Y) = E[XY]-E[X]E[Y]

= E[x(-2x+3)]-E(x)E(-2x+3)

$= E[-2x^2 + 3x] - E[x]E[-2x + 3]$

= -2E[x^2]+3E[x]-E[x][-2E(x)+E(3)]

$= -2E[x^2] + 3E[x] - E[x][-2E[x] + 3]$

$= -2E[x^2] + 2[E(x)]^2$

$= -2[E[x^2] - [E(x)]^2]$

= -2 Var[x]

19. Show that $\boldsymbol{Cov^2(X, Y) \leq Var[X]Var[Y]}$

Cov(X,Y) = E[XY]-E[X]E[Y]

We know that $[E[XY]^2 \leq E(X^2)E(Y^2)$

$$Cov^2(X, Y) = E[XY]^2 + [E(X)]^2[E(Y)]^2 - 2E(XY)E(X)E(Y)$$

$\leq E(X^2)E(Y^2) + \left(E(X^2)\right)\left(E(Y^2)\right) - 2E(XY)E(X)E(Y)$

$\leq E(X^2)E(Y^2) + \left(E(X^2)\right)\left(E(Y^2)\right) - E(X^2)E(Y^2) - E(Y^2)E(X^2)$

$\leq Var(X)Var(Y)$

20. If the pdf of X is $f_X(x) = e^{-x}, x > 0$find the p.d.f of Y=2X+1

Given $f_X(x) = e^{-x}, x > 0$

and Y=2X+1

⇨ $y = 2x + 1$

⇨ $x = \frac{y-1}{2} = f(y)$

$\frac{dx}{dy} = \frac{1}{2}$

$\left|\frac{dx}{dy}\right| = \frac{1}{2}$

$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$

$= e^{-x}\frac{1}{2}, x > 0$

$= e^{-\left(\frac{y-1}{2}\right)}$

$x > 0$ => $\frac{y-1}{2} > 0$

=> $y > 1$

 $f_Y(y) = e^{-\left(\frac{y-1}{2}\right)}, y > 1$

PART-B

1) The joint pdf of a two dimensional random variable (X,Y) is given by f(x,y)=$xy^2+\frac{x^2}{8}$,$0 \leq x \leq 2,\ 0 \leq y \leq 1$. compute i) p(x>1/y<$\frac{1}{2}$) ii) p(y<$\frac{1}{2}$/x>1) iii) p(x<y) iv) p(x+y$\leq$ 1).

2) The joint probability density function of the two variable is $f(x,y)=\begin{cases} \frac{8}{9}xy, & 1 < x < y < 2 \\ 0, & otherwise \end{cases}$.

i) Find the marginal density functions of X and Y.

ii) Find the conditional density function of Y given X=x.

3) The joint pdf of R.V. X and Y is given by $f(x,y)=kxye^{-(x^2+y^2)}, x > 0, y > 0$. Find the value of k and prove also that X and Y are independent.

4) Calculate the correlation coefficient for the following data.

X	65	66	67	67	68	69	70	72
Y	67	68	65	68	72	72	69	71

5) Two random variables X and Y having the following joint p.d.f $f(x,y)=\begin{cases} 2 - x - y, & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & otherwise \end{cases}$. Find the correlation coefficient between X and Y.

6) If the joint p.d.f of (X, Y) is given by $f_{xy}(x,y) = x + y,\ 0 \le x, y \le 1$. Find the p.d.f of U=XY.

7) If the joint density of $X_1 and\ X_2$ is given by $f(x_1, x_2) = \begin{cases} 6e^{-3x_1-2x_2}, & x_1 > 0, x_2 > 0 \\ 0, & otherwise \end{cases}$.

Find the probability density of $Y = X_1 + X_2$.

8) Let $X_1, X_2, \dots, X_{100}$ be independent identically distributed random variables with μ=2, and $\sigma^2 = \frac{1}{4}$. Find $p(192 < X_1 + X_2 + \dots + X_{100} < 210)$.

UNIT III

MARKOV PROCESSES AND MARKOV CHAINS

1. **Give an example for a continuous time random process.**

   If both T and S are continuous, the random process is called a continuous time random process. For example if X(t) represents the maximum temperature at a place in the interval(0,t),{X(t)} is a continuous random process.

2. **State the four types of a stochastic processes.**
   (i) Discrete time, discrete state random process
   (ii) Discrete time, continuous state random process
   (iii) Continuous time, discrete state random process
   (iv) continuous time, continuous state random process

3. **Define discrete random process. Give an example.**

   If X is discrete and t is continuous ,the random process is called as discrete random process.

   <u>Example</u>

   If X(t) represents the number of telephone calls received in the interval (0,t) then {X(t)} is a discrete random process since S={0,1,2,3,….}

4. **Define a stationary process**.

   If certain probability distribution or averages do not depend on t, then the random process is called stationary. A random process is called strongly stationary process or strict sense stationary process (SSS), if all its finite dimensional distributions are invariant under translation of time parameter.

5. **Define Wide sense stationary random process.**

   A random process {X(t)} is said to be wide sense stationary if its mean is constant and its autocorrelation depends only on time deference.

   i.e., (i) E(X(t)) = constant

   (ii)$R_{XX}(t, t+\tau) = R_{XX}(\tau)$

6. **When is a random process said to be ergodic**.

   All the time averages are equal to the corresponding ensemble averages.

7. **Give an example of an ergodic process.**
   (i) A markov chain with finite state space
   (ii) A stochastic process X(t) is ergodic if its time average tends to the ensemble average as $t \to \infty$

8. **Define evolutionary process.**

A random process {X(t)} that is not stationary in any sense is called an evolutionary process.

9. **Define Markov process.**

If for $t_1 < t_2 < \cdots < t_n < t$,

$$P\{X(t) \leq x \ / X(t_1) = x_1, X(t_2) = x_2, \dots \dots . X(t_n) = x_n\}$$
$$= P\{X(t) < x \backslash X(t_n) = x_n\}$$

Then the process {X(t)} is called as markov chain

10. **Give an example of Markov process**

Let X(t)=number of births upto time 't' so that the sequence {X(t),$t \epsilon [0, \infty)$}forms a pure birth process. Then it forms a Markov process since the future is independent of the past given the current state.

11. **Define Markov chain and one step transition probability.**

Let X(t) be a markov process which posses Markov property which takes only discrete values whether t is discrete or continuous is called as Markov chain.

When an experiment is conducted the state of Markov chain are denoted by $(a_1, a_2, \dots \dots, a_n)$.The one step transition probability is defined by the conditional probability that the n^{th}step trial of reaching the state$a_j$from the state a_i are denoted by $P_{ij}[n-1, n] = [P\{x_n = a_j \backslash x_{n-1} = a_i\}]$

12. **When you say that a Markov chain is homogeneous.**

If the one step transition probability does not depend on the step, i.e.,$P_{ij}(n-1, n) = P_{ij}(m-1, m)$the markov chain is called a homogeneous markov chain.

13. **Define irreducible markov chain. And state Chapman-Kolmogorov theorem.**

If $P_{ij}(n) > 0$ for some n and for all i & j ,then every state can be reached from every other state. When this condition is satisfied, the Markov chain is said to be irreducible. The tpm of an irreducible chain is an irreducible matrix.

Chapman-Kolmogorov theorem

If P is the tpm of a homogeneous Markov chain , then the n-step tpm P(n) is equal to P^n i.e.,$[P_{ij}(n)] = [P_{ij}]^n$

14. Determine whether the given matrix is irreducible or not $\boldsymbol{P} = \begin{bmatrix} 0.3 & 0.7 & 0 \\ 0.1 & 0.4 & 0.5 \\ 0 & 0.2 & 0.8 \end{bmatrix}$

$$P^2 = P \times P = \begin{vmatrix} 0.3 & 0.7 & 0 \\ 0.1 & 0.4 & 0.5 \\ 0 & 0.2 & 0.8 \end{vmatrix} \begin{vmatrix} 0.3 & 0.7 & 0 \\ 0.1 & 0.4 & 0.5 \\ 0 & 0.2 & 0.8 \end{vmatrix}$$

$$= \begin{vmatrix} 0.16 & 0.49 & 0.35 \\ 0.07 & 0.33 & 0.60 \\ 0.02 & 0.24 & 0.74 \end{vmatrix}$$

 $All\ P_{ij}(2) > 0$. Hence P is irreducible.

15. The one-step transition probability of a Markov chain with states (0,1) is given b y $\boldsymbol{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$**.Is it irreducible markov chain?**

$$Given\ P = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$P^2 = P.P = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$P_{00}^{(2)} = 1 > 0$, $P_{11}^{(2)} = 1 > 0$; $P_{01}^{(1)} = 1 > 0$; $P_{10}^{(1)} = 1 > 0$

 $P_{ij}^{(n)} > 0\ for\ all\ i,j = 0,1\ and\ n = 1,2,3,\ldots$

Hence the markov chain is irreducible.

16. If the tpm of a Markov chain is $\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$**find the steady-state distribution of the chain.**

If $\pi = (\pi_1, \pi_2)$ is the steady state distribution of the chain ,then by the property of π,we have $\pi P = \pi$ where $\pi = (\pi_1, \pi_2)$ and $(\pi_1 + \pi_2) = 1$

 $(\pi_1, \pi_2) \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (\pi_1, \pi_2)$

$\left(\frac{1}{2}\pi_2 \quad \pi_1 + \frac{1}{2}\pi_2\right) = (\pi_1, \pi_2)$

$\frac{1}{2}\pi_2 = \pi_1 \Rightarrow 2\pi_1 - \pi_2 = 0$ ……………………(1)

Similarly $\pi_1 + \frac{1}{2}\pi_2 = \pi_2$

$\Rightarrow 2\pi_1 - \pi_2 = 0$…………………(2)

Also $(\pi_1 + \pi_2) = 1$………………………..(3)

Solving (1) and (3),we get $\pi_1 = \frac{1}{3}$ *and* $\pi_2 = \frac{2}{3}$

Hence the steady-state probability distribution is $\begin{pmatrix}\frac{1}{3} & \frac{2}{3}\end{pmatrix}$

17. Define Poisson process.Is it a stationary process.

If X(t) represents the number of occurrences of a certain event in (0,t), then the discrete random process {X(t)} is called the poisson process , provided that the following postulates are satisfied.

(i) P[1 occurrence in $(t, t + \Delta t) = \lambda\Delta t + o(\Delta t)$

(ii) P[o occurrence in $(t, t + \Delta t) = 1 - \lambda\Delta t + o(\Delta t)$

(iii) P[2 or more occurrences in $(t, t + \Delta t) = o(\Delta t)$

(iv) X(t) is independent of the no. of occurrences of the event in any interval prior or after the interval (0,t)

(v) The probability that the event occurs a specified number of times in $(t_{0\,,} t_0 + t)$ depends on t ,but not on t_0

Poisson process is not a stationary process , as its statistical characteristics are time dependent.

18. What will be the superposition of 'n' independent poisson processes with respective average rates $\lambda_1, \lambda_2, \dots, \lambda_n$.

the superposition of 'n' independent poisson processes with respective average rates $\lambda_1, \lambda_2, \dots, \lambda_n$ is another poisson process with average rate $\lambda_1 + \lambda_2 + \cdots + \lambda_n$

19. State any four properties of Poisson process.

(i) The Poisson process is a Markov process

(ii) The sum of two independent Poisson processes is again a Poisson process

(iii) The difference of two independent poisson processes is not a poisson process

(iv) The inter-arrival of a poisson process with parameter λ has an exponential distribution with mean 1/λ

20. If the customers arrive in accordance with a Poisson process with a mean rate of 2 per minute, find the probability that the interval between two consecutive arrivals is more than one minute.

The interval T between 2 consecutive arrivals follows an exponential distribution with parameter λ=2, then

$f(t) = \lambda e^{-\lambda t}$

$P\{T > 1\} = \int_1^\infty 2e^{-2t}dt = e^{-2} = 0.135$

PART-B

1) Show that the process {X (t)}, X (t) =ACos (ωt+θ) where A, ω are constant, θ is uniformly distributed in (-π, π) is wide sense stationary.

2) Show that the process X (t) =Acosλt+Bsinλt is wide sense stationary A and B are Random Variables if i) E (A)=E(B)=0. ii) E (A^2) =E (B^2). iii) E (AB) =0.

3) The process {X (t)} whose probability distribution under certain conditions is given by

p{X(t)=n}=$\begin{cases} \frac{(at)^{n-1}}{(1+at)^{n+1}}, & n = 1,2,3,\ldots \\ \frac{at}{1+at}, & n = 0 \end{cases}$. Show that it is not stationary.

4) Consider the random process {X (t)} with X (t) = Acos (A^2t +ϕ). Where ϕ is a uniformly distributed random variable in (-π, π).prove that {x (t)} is correlation ergodic.

5) The transition probability matrix of a Markov chain $\{X_n\}$, n=1, 2, 3,… having 3 states 1,2 and 3 is P=$\begin{matrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{matrix}$ and the initial distribution is $P^{(0)}$=(0.7 0.2 0.1).

Find i) P $\{X_2 = 3\}$ ii) P $\{X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2\}$.

6) Derive the probability law for the Poisson process {X (t)}.

7) If customers arrive at a counter in accordance with a Poisson process with a mean rate of 3 per minute, Find the probability that the interval between 2 consecutive arrivals is

i) More than 1 minute

ii) Between 1 minute and 2 minutes

iii) 4 minutes or less

8) If {X (t)} and {Y(t)} are two independent Poisson process, show that the conditional distribution of {X(t)} given {X(t)+Y(t)} is binomial

UNIT IV

QUEUEING THEORY

1. **What are the basic characteristics of a queueing system .**

The basic characteristics of a queueing system are

(i) Arrival pattern of customers

(ii) Service pattern of servers

(iii) Queue discipline

(iv) System capacity

2. **What are the basic characteristics of a queueing process.**

The basic queueing process describes how customers arrive and proceed through the queueing system. This means that the basic queueing process describes the operation of a queueing system.

(a) the calling population

(b) the arrival process

(c) the queue configuration

(d) the queue discipline

(e) the service mechanism

3. **Define queue discipline**

This is the procedure by which customers are selected for service when a queue has formed.

In general, properties of a queueing system which are concerned with waiting time depend on queue discipline.

The various types of queue discipline are

(i) FIFO or FCFS → First in First Out or First Come First Served.

This is the most commonly used procedure in servicing customers.

(ii) LIFO or LCFS → Last in First Out or Last Come First Served

This procedure is used in inventory systems.

(iii) SIRO → Selection for Service In Random Order.

(iv) PIR → Priority in selection

4. For (M/M/1):(∞/FIFO) model , write the Little's formula.

(i) $L_s = \frac{\lambda}{\mu-\lambda} = \boldsymbol{\lambda W_s} = \boldsymbol{L_q} + \frac{\lambda}{\mu}$

(ii) $W_s = \frac{1}{\mu-\lambda} = W_q + \frac{1}{\mu}$

(iii) $L_q = \frac{\lambda^2}{\mu-\lambda} = \boldsymbol{\lambda W_q}$

(iv) $W_q = \frac{\lambda}{\mu(\mu-\lambda)}$

5. Give the formulas for the waiting time of a customer in the queue and in the system for the (M/M/1):(∞/FIFO) model

(i) Waiting time of a customer in the queue $W_q = \frac{\lambda}{\mu(\mu-\lambda)}$

(ii) Waiting time of a customer in the system $W_s = \frac{1}{\mu-\lambda}$

6. Write down the probability density function of the waiting time of a customer in the (M/M/1):(∞/FIFO) queue system.

The probability density function of the waiting time of a customer in the (M/M/1):(∞/FIFO) queue system is $f(w) = (\mu - \lambda)e^{-(\mu-\lambda)w}, w \geq 0$

7. In the usual notation of a (M/M/1):(∞/FIFO) queue system find P(N>2) if λ=12/hr and μ=30/hr.

Given λ=12/hr

μ=30/hr.

$\rho = \frac{\lambda}{\mu} = \frac{12}{30} = \frac{2}{5}$

We know that $P(N > k) = \rho^{k+1}$

$$P(N > 2) = \left(\frac{2}{5}\right)^3 = 0.064$$

8. **What is the probability that a customer has to wait more than 15 minutes to get his service completed in (M/M/1):(∞/FIFO) queue system if $\boldsymbol{\lambda}$=6/hr and μ=10/hr**.

The probability that the waiting time of a customer in the system exceeds $t = e^{-(\mu-\lambda)t}$

Given $\boldsymbol{\lambda}$=6/hr

μ=10/hr, t=15min =1/4 hr

$\therefore$ $The\ required\ probability = e^{-(10-6)\frac{1}{4}} = e^{-1} = 0.3679$

9. **In a given (M/M/1):(∞/FIFO) queue, ρ=0.6. What is the probability that the queue contains 5 or more customers.**

The probability that the queue contains 5 or more customers is given by $P(N \geq 5) = \rho^5 = (0.6)^5 = 0.0778$

10. **Write down the formula for $\boldsymbol{P_n}$ interms of $\boldsymbol{P_0}$for the (M/M/s):(∞/FIFO) queueing system.**

$$P_0 = \frac{1}{\left|\sum_{n=0}^{s-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!\left(1-\frac{\lambda}{\mu s}\right)}\left(\frac{\lambda}{\mu}\right)^s\right]}$$

$$P_n = \frac{1}{s!s^{n-s}}\left(\frac{\lambda}{\mu}\right)^n P_0$$

11. **Find $\boldsymbol{P(X \geq c + n)}$ for an M/M/c queueing system**.

$$P(X \geq c + n) = \frac{\left(\frac{\lambda}{\mu}\right)^c\left(\frac{\lambda}{c\mu}\right)^k}{c!\left(1-\frac{\lambda}{c\mu}\right)}P_0$$

where $P_0 = \left|\sum_{n=0}^{c-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{\left(\frac{\lambda}{\mu}\right)^c}{c!} \frac{1}{\left(1-\frac{\lambda}{c\mu}\right)}\right|^{-1}$

12. Define effective arrival rate with respect to an (M/M/1):(k/FIFO) queueing model

The effective arrival rate is denoted by λ' and is defined by $\lambda' = \mu(1 - P_0)$

13. Give the probability that there is no customer in an (M/M/1):(k/FIFO) queueing model.

Probability that there is no customer in the system is $P_0 = \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}$

14. Given that λ=6/hr and μ=4/hr , what is the average waiting of a customer in the queue for the model (M/M/3):(∞/FIFO) if he happens to wait.

Given c=3

λ=6/hr; μ=4/hr

$$E\lfloor W_q / W_s > 0\rfloor = \frac{1}{\mu c-\lambda} = \frac{1}{(4)(3)-6} = \frac{1}{6}/hr = \frac{1}{6} \times 60min = 10min$$

15. In (M/M/1):(k/FIFO) model if λ=3/hr ,μ=4/hr ,$P_0 = 0.28$. What is the effective arrival rate of a customer?

Given λ=3/hr, μ=4/hr,$P_0 = 0.28$.

Effective arrival rate $\lambda' = \mu(1 - P_0) = 4(1 - 0.28) = 2.88/hr$

16. In a 3 server infinite capacity poisson queue model if $\frac{\lambda}{c\mu} = \frac{2}{3}$ $find\ P_0$

Given $\frac{\lambda}{c\mu} = \frac{2}{3}$

No of servers c=3

 $\frac{\lambda}{3\mu} = \frac{2}{3} => \frac{\lambda}{\mu} = \frac{2}{3} \times 3$

$=> \frac{\lambda}{\mu} = 2$

We know that $P_0 = \left[\sum_{n=0}^{c-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{\left(\frac{\lambda}{\mu}\right)^c}{c!} \frac{1}{\left(1-\frac{\lambda}{c\mu}\right)}\right]^{-1}$

$$P_0 = \left[\sum_{n=0}^{2} \frac{1}{n!}(2)^n + \frac{(2)^3}{3!} \frac{1}{\left(1-\frac{2}{3}\right)}\right]^{-1} = \frac{1}{9}$$

17. Suppose there are 2 parallel servers. Each can serve an average of 6 customers /hr. If λ =10/hr and $P_0 = \frac{1}{11}$ find the number of customers in the queue.

Given λ=10/hr, μ=6/hr,$P_0 = \frac{1}{11}$.

No of servers c=2

We know that, average number of customers in the queue is

$L_q = \frac{1}{(c-1)!}\left(\frac{\lambda}{\mu}\right)^c \frac{\lambda\mu}{(c\mu-\lambda)^2} P_0$

$= \frac{1}{(1)!}\left(\frac{10}{6}\right)^2 \frac{(10\times6)}{(2\times6-10)^2} \frac{1}{11} = \left(\frac{5}{3}\right)^2 \frac{60}{(12-10)^2} \frac{1}{11} = 3.78$

$L_q \approx 4$ Customers

18. In a single server and capacity 20 model with Poisson arrival rate 4/hr and expected service rate 6/hr , what is the effective arrival rate and percentage of idle time.

Given λ=4/hr, μ=6/hr

N=20

$\rho = \frac{\lambda}{\mu} = \frac{4}{6} = \frac{2}{3}$

Effective arrival rate, $\lambda' = \mu(1 - P_0)$

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-\frac{2}{3}}{1-\frac{2}{3}^{21}} = 0.33$$

$$\lambda' = 6(1 - 0.33) = 3.99$$

Percentage of idle time =$P_0 \times 100$

= 0.33x100=33%

19. What is the probability that an arrival to an infinite capacity 3 server Poisson queueing system with $\frac{\lambda}{\mu} = 2$ and $P_0 = \frac{1}{9}$ enters the service without waiting.

P (without waiting) =P (N <3)

$$= P_0 + P_1 + P_2$$

$$P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0 \ when\, n \le c = 3$$

$$P(N<3) = \frac{1}{9} + \frac{2}{9} + \frac{1}{2} \times 2^2 \times \frac{1}{9} = \frac{5}{9}$$

20. State the formula for average number of customers in the (M/M/C):(k/FCFS)

$$L_q = P_0\left(\frac{\lambda}{\mu}\right)^C \frac{\rho}{C!(1-\rho)^2}[1 - \rho^{k-C} - (k-C)(1-\rho)\rho^{k-C}]$$

PART-B

1. If people arrive to purchase cinema tickets at the average rate of 6 per minute, it takes an average of 7.5 seconds to purchase a ticket. If a person arrives 2 min before the picture starts and if it takes exactly 1.5 min to reach the correct seat after purchasing the ticket.
   (i) Can he expect to be seated for the start of the picture?
   (ii) What is the probability that he will be seated for the start of the picture?
   (iii) How early must he arrive in order to be 99% of sure of being seated for the start of the picture?
2. Customers arriving at a watch repair shop according to Poisson process at a rate of one per every 10 minutes and the service time are an exponential random variable with mean 8 minutes.
   (i) Find the average number of customers L_s in the shop.
   (ii) Find the average time a customer spends in the shop W_s.
   (iii) Find the average number of customer in the queue L_q.
   (iv) What is the probability that the server is idle?
3. There are 3 typists in an office. Each typist can type an average of 6 letters per hour. If letters arrive for being typed at the rate of 15 letters per hour.
   (i) What fraction of the time all the typists will be busy?
   (ii) What is the average number of letters waiting to be typed?
   (iii) What is the average time a letter has to spend for waiting and for being typed?

(iv) What is the probability that a letter will take longer than 20 minutes waiting to be typed and being typed?

4. A petrol pump station has 4 pumps. The service time follow the exponential distribution with a mean of 6 minutes and cars arrive for service in a Poisson process at the rate of 30 cars per hour.
   (i) What is the probability that an arrival would have to wait in line?
   (ii) Find the average waiting time, average times spend in the system and the average number of cars in the system.
   (iii) For what % of time would a pump be idle on an average?
5. Patients arrive at a clinic according to Poisson distribution at a rate of 30 patients per hour. The waiting room does not accommodate more than 14 patients. Examination time per patient is exponential with mean rate of 20 per hour.
   (i) Find the effective arrival rate at the clinic.
   (ii) What is the probability that an arriving patient will not wait?
   (iii) What is the expected waiting time until a patient is discharged from the clinic?
6. The local one person barber shop can accommodate a maximum of 5 people at a time (4 waiting and 1 getting hair-cut). Customers arrive according to a Poisson distribution with mean 5/hr. the barber cuts hair at an average rate of 4/hr. (Exponential service time).
   (i) What % of time is the barber idle?
   (ii) What fractions of the potential customers are turned away?
   (iii) What is the expected number of customers waiting for a hair-cut?
   (iv) How much time can a customer expect to spend in the barber shop?
7. A 2 person barber shop has 5 chairs to accommodate waiting customers. Potential customers, who arrive when all 5 chairs are full, leave without entering barber shop. Customers arrive at the average rate of 4 per hour and spend an average of 12 minutes in the barber's chain. Compute $P_0, P_1, P_7, E(N_q)$ and $E(W)$.
8. At a port there are 6 unloading berths and 4 unloading crews. When all the berths are full, arriving ships are diverted to an overflow facility 20 kms down the river. Tankers arrive according to a Poisson process with a mean of 1 every 2 h. it takes for an

unloading crew, on the average, 10 h to unload a tanker, the unloading time following an exponential distribution. Find

(a) How many tankers are at the port on the average?

(b) How long does a tanker spend at the port on the average?

(c) What is the average arrival rate at the overflow facility?

Unit V

NON MARKOVIAN QUEUES AND QUEUE NETWORKS

1. Write down the Pollaczek-Khinchine formula and explain the notations.

If T is the random service time, the average number of customers in the system $L_s = E_n = \lambda E(T) + \frac{\lambda^2[VAR(T)+E(T)^2]}{2[1-\lambda E(T)]}$ where E (T) is the mean of T and V(T) is variance of T.

2. M/G/1 queueing system is Markovian. Comment on this statement.

M/G/1 queueing system is a non-Markovian queue model, since the service time follows general distribution.

3. Write down the Pollaczek-Khinchine transform formula

$$V(s) = \frac{(1-\rho)(1-s)B^{*}(\lambda-\lambda_s)}{B^{*}(\lambda-\lambda_s)-s}$$

4. In (M/G/1) model write down the formula for the average number of customers in the system.

$$W_s = \frac{\lambda^2\sigma^2+\rho^2}{2\lambda(1-\rho)} + \frac{1}{\mu}$$

5. Write classification of Queueing networks.

The classification of Queueing networks are

(i) Open networks

(ii) Closed networks

(iii) Mixed networks

6. **State arrival theorem.**

In the closed network system with m customers, the system as seen by arrivals to server j is distributed as the stationary distribution in the same network system when there are only m-1 customers.

7. **What do you mean by M/G/1 queue?**

In the M/G/1 Queueing system under study, we consider a single-server queueing system with infinite capacity, Poisson arrivals and general service discipline. This model has arbitrary service time, and it is not necessary to be memory less. i.e., it is not exponential.

8. **What is meant by queue network?**

A network of queues is a collection of service centers, which represent system resources, and customers, which represent users or transactions.

9. **What is meant by bottleneck of a network?**

As the arrival rate λ in a 2-state tandem queue model increases, the node with the larger value of $\rho_i = \frac{\lambda}{\mu_i}$ will introduce instability. Hence the node with the larger value of ρ_i is called the bottleneck of the system.

10. **In an M/G/1 Model , if $\lambda = 5min, \mu = 6\,\text{min}, and\ \sigma = \frac{1}{20}$ Find the value of L_s**

Given $\lambda = 5min, \mu = 6\,\text{min}, and\ \sigma = \frac{1}{20}$

$\rho = \frac{\lambda}{\mu} = \frac{5}{6}$

We know that $L_s = \rho + \frac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)}$

$$= \frac{5}{6} + \frac{5^2\left(\frac{1}{20}\right)^2+\left(\frac{5}{6}\right)^2}{2\left(1-\frac{5}{6}\right)} = \frac{5}{6} + \frac{\frac{1}{16}+\frac{25}{36}}{2\left(\frac{1}{6}\right)} = 3.103$$

$L_s = 3.103$

11. **Consider a service facility with sequential stations with respective service rates of 3/min and 4/min. The arrival rate is 2/min. What is the average service time of the system, if the system could be approximated by a two stage tandem queue?**

Given $\lambda = 2, \mu_1 = 3, \mu_2 = 4$

The average service time of the system $= \frac{1}{\mu_1 - \lambda} + \frac{1}{\mu_2 - \lambda}$

$= \frac{1}{3-2} + \frac{1}{4-2} = 1 + \frac{1}{2} = \frac{3}{2}$ / min

12. **Define closed queueing network.**

In a closed queueing network, jobs neither enter nor depart from the network. If the network has multiple job classes then it must be closed for each class of jobs.

13. **Define open queueing network.**

An open queueing network is characterized by one or more sources of job arrivals and corresponding one or more sinks that absorb jobs departing from the network. If the network has multiple job classes the it must be open for each class of jobs.

14. **Define a two stage tandem queues.**

Consider a two-server system In which customers arrive at a Poisson rate λ at server 1. After being served by server 1 they then join the queue in front of server 2. We suppose there is infinite waiting space at both servers. Each server serves one customer at a time with server i taking an exponential time with rate μ_i for a service, i=1, 2,3,…. Such a system is called a tandem or sequential system.

15. **Write down the (flow balance) traffic equations for an open Jackson network.**

Jacksons flow balance equations for this open model are $\lambda_j = r_j + \sum_{i=1}^{k} \lambda_i P_{ij}, j = 1,2, \ldots \ldots, k$

16. **Write down the (flow balance)traffic equations for a closed Jackson network**

Jacksons flow balance equations for this closed model are $\lambda_j = \sum_{i=1}^{k} \lambda_i P_{ij}, j = 1,2, \ldots \ldots, k$

17. **Write down the formula for the steady state probability $P(n_1, n_2, \ldots, n_k)$ for multiple servers Jackson's closed network.**

$$P(n_1, n_2, \dots, n_k) = D_N\, \rho_1^{n_1}\rho_2^{n_2}\rho_3^{n_3} \text{where } N = n_1 + n_2 + \cdots + n_k$$

$$\text{and } D_N = \sum_{n_1+n_2+\cdots+n_k} \rho_1^{n_1}\rho_2^{n_2} \dots\dots \rho_k^{n_k} \quad \text{and } \rho_j = \frac{\lambda_j}{\mu_j}$$

18. Write the formula for the steady state joint probability for m and n customers in the nodes S_1 and S_2 respectively for a 2-stage series queue model.

$$P(m,n) = \left(\frac{\lambda}{\mu_1}\right)^m \left(1 - \frac{\lambda}{\mu_1}\right)\left(\frac{\lambda}{\mu_2}\right)^n \left(1 - \frac{\lambda}{\mu_2}\right)$$

19. Write down the characteristics of Jackson networks.

1. Arrivals from outside through node “i” follow a Poisson process with mean arrival rate "r_i"
2. Services times at each channel at node i are independent and exponentially distributed with parameter μ_i
3. The probability that a customer who has completed service at node “ i” will go to the next node “j” is P_{ij} where i=1,2,……,k, j=0,1,2,…….,k. Also P_{i0} denotes the probability that a customer will leave the system from the node i

20. Automatic car wash facility operates with only one bay. Cars arrive according to a Poisson process with mean of 4 cars per hour and may wait in the facility’s parking lot if the bay is busy.If the service times for all cars is constant and equal to 10 minutes. Determine L_s and W_s

Given $\lambda = 4 \text{ per hour} = \frac{4}{60} \text{per minute} = \frac{1}{15}$ /min

Given Service time ‘T’ is constant.

 ‘T’ follows a distribution with E (T) =10 and VAR (T) =0

By Pollaczek-Khinchine formula

$$L_s = \lambda E(T) + \frac{\lambda^2[VAR(T) + E(T)^2]}{2[1 - \lambda E(T)]}$$

$$= \frac{1}{15} \times 10 + \frac{\left(\frac{1}{15}\right)^2 [0 + 10^2]}{2\left(1 - \frac{1}{15} \times 10\right)}$$

$$= \frac{10}{15} + \frac{\frac{100}{225}}{2\left(\frac{5}{15}\right)} = \frac{10}{15} + \frac{100}{225} \times \frac{15}{10} = \frac{4}{3}$$

$$W_s = \frac{L_s}{\lambda} = 15 \times \frac{4}{3} = 20 \text{ minutes}$$

PART – B

1.Customers arrive at a watch repair shop according t a Poisson process at a rate of one per every 10 minutes and the service time is an exponential random variable with mean 8 minutes. Find the average number of customer L_s , the average waiting time a customer spends in the shop W_s and the average time a customer spends in the waiting for service W_q.[A.U model]

2.Find the MGF of the binomial distribution and hence find its mean and variance.[A.U.A/M 2004]

3.Derive Pollaczek – Khintchine formula.

4.Automataic car wash facility operates with only one Bay. Cars arrive according t a poisson process, with mean of 4 cars per hour and may wait in the facility's parking lot if the bay is busy . If the service time for all cars is constant and equal to 10 min, determine L_s, L_q, W_s and W_q [A.U N/D 2005,A/M 2005]

5.Customers arrive at a watch repair shop according to a Poisson process at a rate of one per every 10 minutes, and the service time is an exponential random variable with mean 8 minutes.

1. Find the average number of customers L_s in the shop.
2. Find the average time a customer spends in the shop W_s
3. Find the average number of customers in queue L_q
4. What is the probability that the server is idle.[A.U. A/M 2005]

6.A patient who goes to a single doctor cline for a General check – up has to go through 4 phases. The doctor takes on the average 4 minutes for each phase of the check – up and the time taken for each phase is exponentially distributed of the arrivals of the poisson at the average rate of 3 per hour, what is the average time spent by a patient (i) in the examination? (ii) Waiting in the clinic?

7.A car wash facility operates with only one bay. Cars arrive according to a poisson distribution with a mean of 4 cars per hour and may wait in the facility's parking lot is the bay is busy. The parking lot is large enough to accommodate any number of cars. Find the average number of cars waiting in the packing lot, if the time for washing and cleaning a car follows.

(i) Uniform distribution between 8 and 12 minutes.
(ii) A normal distribution with means 12 minutes and S.D 3 minutes.
(iii) A distribution with values equal to 4, 8 and 15 minutes and corresponding probabilities 0.2, 0.6 and 0.2.