

MARIA COLLEGE OF ENGINEERING AND TECHNOLOGY
NUMERICAL METHODS

UNIT I

SOLUTIONS OF NON LINEAR EQUATIONS

1. Give Newton - Raphson Iterative Formula.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Where $n=0,1,2,\dots$

2. Derive Newton - Raphson Formula to find the cube root of a positive number k .

To find cube root of a positive number k .

$$\text{Let } x = \sqrt[3]{k}$$

$$f(x) = x^3 - k = 0$$

$$f'(x) = 3x^2$$

$$x_{n+1} = x_n - \frac{(x_n^3 - k)}{3x_n^2}$$

$$= \frac{3x_n^3 - x_n^3 + k}{3x_n^2}$$

$$= \frac{1}{3} \left[\frac{2x_n^3 + k}{x_n^2} \right]$$

3. Evaluate $\sqrt{12}$ applying Newton formula?

$$\text{Let } x = \sqrt{12}$$

$$x^2 = 12$$

$$f(x) = x^2 - 12$$

$$f(3) = -ive$$

$$f(4) = +ive$$

Take $x_0 = 3$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 3 - \frac{f(3)}{f'(3)} = 3.5$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 3.5 - \frac{(3.5)^2 - 12}{2(3.5)}$$

∴ The root is 3.4642

4. If $g(x)$ is continuous in $[a, b]$, then under what condition of the iterative method

$x = g(x)$ has a unique solution in $[a, b]$.

$$|g'(x)| < 1 \text{ in } [a, b].$$

5. Find an iterative formula find \sqrt{N} where N is a positive number?

Let $x = \sqrt{N}$

$$x^2 = N$$

$$x^2 - N = 0$$

$$f(x) = x^2 - N$$

$$f'(x) = 2x$$

$$\begin{aligned} \therefore x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^2 - N}{2x_n} \\ &= \frac{2x_n^2 - x_n^2 + N}{2x_n} \\ &= \frac{x_n^2 + N}{2x_n} \end{aligned}$$

$x_{n+1} = \frac{x_n^2 + N}{2x_n}$ is the iteration formula.

6. What is the order of convergence of Newton Raphon method?

The convergence is quadratic and is of order 2.

7. What is the sufficient condition for the convergence in the iteration method?

The sufficient condition for the convergence in the iteration method is $|f'(x)| < 1$ for all x in the interval .

8. Derive Newtons formula to find the cube root of a positive number K .

Let $x = \sqrt[3]{K}$

$$x^3 = K$$

$$x^3 - K = 0$$

$$f(x) = x^3 - K$$

$$f'(x) = 3x^2$$

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{x_n^3 - K}{3x_n^2} \\ &= \frac{3x_n^3 - x_n^3 + K}{3x_n^2} \\ &= \frac{2x_n^3 + K}{3x_n^2} \end{aligned}$$

$x_{n+1} = \frac{2x_n^3 + K}{3x_n^2}$ is the iteration formula.

9. State the formula for the method of False position to determine a root of $f(x) = 0$?

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

10. Find an iterative formula to find the reciprocal of a given number N?

$$\text{Let } x = \frac{1}{N}$$

$$\frac{1}{x} = N$$

$$\frac{1}{x} - N = 0$$

$$f(x) = \frac{1}{x} - N$$

$$f'(x) = -\frac{1}{x^2}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{\frac{1}{x_n} - N}{-\frac{1}{x_n^2}}$$

$$= x_n + x_n^2 \left(\frac{1}{x_n} - N \right)$$

$$= x_n + x_n - Nx_n^2$$

$$= 2x_n - Nx_n^2$$

$x_{n+1} = 2x_n - Nx_n^2$ is the iterative formula

11. Mention any two methods to solve linear algebraic and transcendental equations?

(i) Bisection method

(ii) Regula falsi method

12. Can we apply iteration method to find the root of the equation $2x = \cos x + 3$ in $\left[0, \frac{\pi}{2}\right]$? Justify?

Given $2x = \cos x + 3$

$$x = \frac{\cos x + 3}{2} = g(x)$$

$$g'(x) = \frac{-\sin x}{2}$$

$$g'(0) = \frac{-\sin 0}{2} = -\frac{0}{2} = 0$$

$$|g'(0)| < 1$$

$$g'(\pi/2) = \frac{-\sin \frac{\pi}{2}}{2} = -\frac{1}{2} = -0.5$$

$$\left|g'\left(\frac{\pi}{2}\right)\right| < 1$$

$$\therefore |g'(0)| \text{ and } \left|g'\left(\frac{\pi}{2}\right)\right| < 1$$

Hence we can apply iteration method for $2x = \cos x + 3$ in $\left[0, \frac{\pi}{2}\right]$

13. Establish the formula $x_{i+1} = \frac{1}{2}\left(x_i + \frac{N}{x_i}\right)$

$$\text{Let } x = \sqrt{N}$$

$$x^2 = N$$

$$x^2 - N = 0$$

$$f(x) = x^2 - N$$

$$f'(x) = 2x$$

W.k.t Newton –Raphson method is $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$

$$= x_i - \frac{x_i^2 - N}{2x_i}$$

$$= \frac{2x_i^2 - x_i^2 + N}{2x_i}$$

$$= \frac{x_i^2 + N}{2x_i} = \frac{1}{2}\left(x_i + \frac{N}{x_i}\right)$$

14. When shall we not use Newton –Raphson method?

$|f(x)f''(x)| < |f'(x)|^2$ is not satisfied for all x in the interval, then we cannot use Newton –Raphson method

15. Locate the negative root of $x^3 - 2x + 5 = 0$, approximately?

$$\text{Let } f(x) = x^3 - 2x + 5$$

$$f(-1) = -1 + 2 + 5 = 6(\text{positive})$$

$$f(-2) = -8 + 4 + 5 = 1(\text{positive})$$

$$f(-3) = -27 + 6 + 5 = -16(\text{negative})$$

∴ Root lies between -2 and -3

$$|f(-2)| < |f(-3)|$$

16. Newton's formula converges to

(i) $|f'(x) \cdot f''(x)| < \{f(x)\}^2$

(ii) $|f(x) \cdot f''(x)| < \{f'(x)\}^2$

(iii) $|f(x) \cdot f'(x)| < \{f''(x)\}^2$

(iv) none

(ii) $|f(x) \cdot f''(x)| < \{f'(x)\}^2$

17. Can you find a real root of the equation $\cos x = 3x - 1$ correct to 3 decimal places by using iteration method?

Yes,

Since given $\cos x = 3x - 1$

$$3x = \cos x + 1 \Rightarrow x = \frac{\cos x + 1}{3} = g(x)$$

Let $f(x) = \cos x - 3x + 1$

$$f(0) = \cos 0 + 1 = 1 + 1 = 2$$

$$f(1) = \cos 1 - 3 + 1 = -1.4596$$

The root lies between 0 and 1

$$g'(x) = \frac{-\sin x}{3}$$

$$|g'(0)| = \frac{-\sin 0}{3} = 0 < 1$$

$$|g'(1)| = \left| \frac{-\sin 1}{3} \right| = 0.280 < 1$$

∴ It is possible to find the real root of the given equation.

PART B

1. Solve the positive root of $x - \cos x = 0$ by using bisection method?
2. Find a positive root of $x^3 - 4x - 9 = 0$ correct to three decimal places using bisection method?
3. Solve the positive root by method of tangents of $2x - \log_{10} x = 7$?
4. Using Newton's method, find the root between 0 and 1 of $x^3 = 6x - 4$ correct to five decimal places?
5. Find the positive root of $xe^x = 2$ by False position method?
6. Obtain the positive root of the equation $x^3 - x - 1 = 0$ by the method of False position?
7. Find a root of the equation $x^3 - 4x + 1 = 0$ by using iteration method (Three decimal places)
8. Solve $3x + \sin x = e^x$ using method of iteration?

UNIT II

Solution of simultaneous linear equation

1. State a sufficient condition for Gauss. Jacobi method converge

Let the given equation be

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

The sufficient condition is

$$|a_1| \geq |b_1| + |c_1|$$

$$|b_1| \geq |a_2| + |c_2|$$

$$|c_3| \geq |a_3| + |b_3|$$

2. Write the first iteration values of x, y, z when the equations $27x + 6y - z = 85$,

$6x + 15y + 2z = 72, x + y + 5z = 110$ are solved by Gauss Seidal method.

$$x = \frac{1}{27} [85 - 6y + z] = \frac{85}{27} \quad [\text{putting } y = z = 0]$$

$$y = \frac{1}{15}[72 - 6x - 2z]$$

$$= \frac{1}{15}[72 - 6(85/27)] = \frac{1}{15} \times 53.11 = 3.5407 \quad [\text{putting } z=0]$$

$$z = \frac{1}{5}[110 - x - y]$$

$$z = \frac{1}{5}\left[110 - \frac{85}{27} - 3.5437\right]$$

3. If $g(x)$ is continuous in $[a, b]$, then under what condition the iterative method

$x \equiv g(x)$ has a unique solution in $[a, b]$.

$$|g'(x)| < 1 \text{ in } [a, b].$$

4. Compare Gauss-Jacobi and Gauss-Seidal methods for solving linear system of the form $Ax = B$.

Gauss elimination is direct method

Gauss Seidal is iterative method.

5. State the condition for convergence of Gauss Seidal method ?

Co-efficient matrix should be diagonally dominant.

6. Compare Gauss elimination and Gauss Seidel methods for solving linear systems of the form $AX = B$

Gauss elimination is direct method

Gauss Seidel is indirect method

7. For solving a linear system, compare Gauss elimination and Gauss Jordan method?

In Gauss elimination method the given system is transformed system with upper triangular coefficient matrix which can be solved by back substitution. In Gauss Jordan method coefficient matrix is reduced to an unit matrix and then directly we can find the unknowns.

8. In Gauss elimination method, the coefficient matrix is transformed toform?

Upper triangular form.

9. Write the First iteration values of x, y, z when the equations

$27x + 6y - z = 85; 6x + 15y + 2z = 72; x + y + 5z = 110$ are solved by Gauss Seidel method.

Given $27x + 6y - z = 85; 6x + 15y + 2z = 72; x + y + 5z = 110$

$$x = \frac{1}{27} [85 - 6y + z]$$

$$y = \frac{1}{25} [72 - 6x - 2z]$$

$$z = \frac{1}{5} [110 - x - y]$$

Initial approximation is $x(0) = 0$ and $y(0) = 0$

$$x = \frac{1}{27} [85] = \frac{85}{27}$$

$$y = \frac{1}{25} \left[72 - 6 \frac{85}{27} - 2z \right] =$$

$$z = \frac{1}{5} \left[110 - \frac{85}{27} - y \right]$$

10. Gauss seidle method is better than Gauss Jacobi?

In gauss seidle method a latest value of unknown at each stage of iteration are used in proceeding to the next stage of iteration .Hence the convergence in Gauss seidle method is more rapid than Gauss Jacobian method.

11. Compare Gauss-Jacobi and Gauss-Seidal methods?

Gauss-Jacobi method	Gauss-Seidal method
Convergence rate is slow	The rate of convergence is roughly twice thjat of Gauss-Jacobi
Indirect method	Indirect method
Condition for convergence in the coefficient matrix is diagonally dominant	Condition for convergence in the coefficient matrix is diagonally dominant

12. Is the iterative method a self correcting method always?

In general iteration is a self correcting method since the round off error is smaller.

13. Distinguish between direct and iterative method of solving simultaneous equations?

Direct method involve a certain amount of fixed computations and they are exact solutions .Iterative or indirect methods are those in which is the solution is got by successive approximation .But the method of iteration is not applicable to all system of equations.

14.In an iterative method the amount of computation depends on the degree of accuracy required state true or false?

True

15.Pickup the correct answer?

As soon as the new value for a variable is found by iteration it is used immediately in the following equations .This method is called

(a) Gauss-seidal (b) Jacobi's (c) Gauss-Jorden (d)Relaxation.

Gauss-Seidal

16.State true or false?

The convergence in the Gauss –Seidal method in the thrice as fast as in Jacobi's method?

The statement is false .The rate of convergence of Gauss-Seidal method is roughly twice that of Gauss-Jacobi.

17. Iteration method is a.....method?

Indirect or successive approximation.

PART B

1.Solve by Gauss Jordan method the inverse of the matrix $\begin{vmatrix} 2 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{vmatrix}$

2.Solve by Gauss Jordan method $x - 3y - z = -30$; $2x - y - 2z = 5$; $5x - y - 2z = 142$.

3.By Gauss Elimination method ,find the inverse of $A = \begin{vmatrix} 1 & 1 \\ 2 & 3 \\ 4 & 9 \end{vmatrix}$

4.Solve the system of equation by Gauss Elimination method $x + 2y + z = 3$; $2x + 3y + 3z = 10$; $3x - y + 2z = 13$.

5.Using Gauss Jacobi method ,solve the equations

$2x - 3y + 20z = 25$; $20x + y - 2z = 17$; $3x + 20y - z = -18$.

6.Solve by Gauss Seidel method ,the following system

$$y - x + 10z = 35.61; \quad x + z + 10y = 20.08; \quad y - z + 10x = 11.19$$

$$7. \text{By the method of triangularisation solve } 5x - 2y + z = 4; \quad 7x + y - 5z = 8; \quad 3x + 7y + 4z = 10$$

Unit : 2

Interpolation and approximation

1 . What is the Lagrange's Formula to find 'y' if three sets of values (x_0, y_0) , (x_1, y_1) and (x_2, y_2) are given ?

$$y=f(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2$$

2. Find the second degree polynomial fitting the following data

x	1	2	4
y	4	5	13

$$x_0=1, x_1=2, x_2=4$$

$$y_0=4, y_1=5, y_2=13$$

By Lagrange's formula,

$$\begin{aligned}
 y=f(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2 \\
 &= \frac{(x-2)(x-4)}{(-1)(-3)}(4) + \frac{(x-1)(x-4)}{1(2)}(5) + \frac{(x-1)(x-2)}{(3)(2)}(13)
 \end{aligned}$$

$$\begin{aligned}
&= \frac{(x^2 - 6x + 8)}{3}(4) + \frac{x^2 - 5x + 4}{-2}(5) + \frac{x^2 - 3x + 2}{6}(13) \\
&= \frac{1}{6} [8x^2 - 48x + 64 - 15x^2 + 75x - 60 + 13x^2 - 39x + 26] \\
&= \frac{1}{6} [6x^2 - 12x + 30] \\
f(x) &= x^2 - 2x + 5
\end{aligned}$$

3. Give inverse Lagrange's interpolation formula

$$\begin{aligned}
x = f(y) &= \frac{(y - y_1)(y - y_2) \dots (y - y_n)}{(y_0 - y_1)(y_0 - y_2) \dots (y_0 - y_n)} x_0 + \frac{(y - y_0)(y - y_2) \dots (y - y_{n-1})}{(y_1 - y_0)(y_1 - y_2) \dots (y_1 - y_n)} x_1 + \dots + \\
&\frac{(y - y_0)(y - y_1) \dots (y - y_{n-1})}{(y_n - y_0)(y_n - y_1) \dots (y_n - y_{n-1})} x_n
\end{aligned}$$

4. Give the Newton's divided difference interpolation formula?

$$f(x) = y_0 + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + \dots$$

5. Obtain the divided difference table for the following data.

x	-1	0	2	3
y	-8	3	1	12

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
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-1	-8	11		
0	3	-1	-4	<u>2</u>
2	1	11	4	
3	12			

6. From the divided difference table for the following data :

x	2	5	10
y	5	29	109

x	y	Δy
5	8	
29	16	1
109		

7. Find the polynomial which takes the following values.

x	0	1	2
y	1	2	1

x	y	Δy	$\Delta^2 y$
0	1		
1	2	1	
2	1	-1	-2

The Newton's Forward interpolation formula is $y(x_0+nh) = y_0 + n \Delta y_0 + \frac{n(n-1)}{2!} \Delta^2 y_0 + \dots$

Here $x_0+nh = x$

$x_0=1, y_0= 1, h = 1$ and $n = x$

$$y(x) = 1 + x(1) + \frac{x(x-1)}{2!}(-2)$$

$$= 1+x-x^2+x$$

$$y(x) = 1+2x-x^2$$

8. Obtain the interpolation quadratic polynomial for the given data by using Newton's forward difference formula

x	0	2	4	6
y	-3	5	21	45

x	f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$
0	-3	8		
2	5	16	8	
4	21	24	8	0
6	45			

$$x_0 = 0, y_0 = -3, h = 2$$

The Newton's forward Interpolation formula is,

$$y(x_0 + nh) = y_0 + n \Delta y_0 + \frac{n(n-1)}{2!} \Delta^2 y_0 + \dots$$

$$x_0 + nh = x$$

$$0 + n(2) = x$$

$$n = x/2$$

$$y(x) = -3 + \frac{x}{2}(8) + \frac{\left(\frac{x}{2}\right)\left(\frac{x}{2}-1\right)}{2!}(8) + \frac{\left(\frac{x}{2}\right)\left(\frac{x}{2}-1\right)\left(\frac{x}{2}-2\right)}{2!}(0) + \dots$$

$$= -3 + 4x + \left(\frac{x}{2}\right)\left(\frac{x}{2}-1\right)(4)$$

$$y(x) = x^2 + 2x - 3$$

9. State Newton's formula on interpolation.

Forward Formula

$$y(x_0 + nh) = y_0 + n \Delta y_0 + \frac{n(n-1)}{2!} \Delta^2 y_0 + \dots$$

Backward Formula.

$$y(x_0 + nh) = y_0 + n \nabla y_0 + \frac{n(n-1)}{2!} \nabla^2 y_0 + \dots$$

10. Given $y_0=2, y_1=4, y_2=8, y_4=32$, find $y_3=?$

$$y_0 = 2$$

$$y_1 = 4 = 2^2$$

$$y_2 = 8 = 2^3$$

$$y_3 = 16 = 2^4$$

$$y_4 = 32 = 2^5$$

11. Find the divided difference table for the following.

x	0	1	4	5
f(x)	8	11	78	123

x	f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$
0	8	3		
1	11	<u>67</u>	4.83	
4	78	3	5.67	
5	123	45		0.168

11. Find the difference table for the following.

x	5	6	9	11
f(x)	12	13	15	18

x	y	Δy_0	$\Delta^2 y$	$\Delta^3 y$
5	12	1		
6	13	2	1	
9	15	3	1	0
11	18			

12. Write the divided difference table for

x	30	35	45	55
y	148	96	68	34

x	y	Δy	$\Delta^2 y$
30	148	$\frac{96 - 148}{35 - 30} = -10.4$	
35	96		$\frac{-2.8 + 10.4}{45 - 30} = 0.506$
45	68	$\frac{68 - 96}{45 - 35} = -2.8$	
55	34	$\frac{34 - 68}{55 - 45} = -3.4$	$\frac{-3.4 + 2.8}{55 - 35} = -0.03$

13. A third degree polynomial passer through (0,1), (1,-1), (2,-1) and (3,2).

Find its value at x=4.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		-2		
1	-1		2	
		0		1
2	-1		3	
		3		
3	2			

The Newton's forward formula is

$$y(x_0 + nh) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!} \Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!} \Delta^3 y_0 + \dots$$

Here $x_0 + nh = x$

Given $x_0 = 0, y = 1, h = 1$

$\therefore n = x$

$$\therefore y(x) = 1 + x(-2) + \frac{x(x-1)}{2!} (2) + \frac{x(x-1)(x-2)}{3!} (1)$$

$$= 1 - 2x + x(x-1) + \frac{x(x-1)(x-2)}{6}$$

$$Y(4) = 1 - 2(4) + 4(4-1) + \frac{4(3)(2)}{6}$$

$$= 1 - 8 + 12 + 4$$

$$= 9$$

14. Construct a linear interpolating polynomial given the points (x_0, y_0) and (x_1, y_1) .

$$y(x) = \frac{(x - x_1)}{(x_0 - x_1)} y_0 + \frac{(x - x_0)}{(x_1 - x_0)} y_1$$

15. Newton's divided difference formula for unequal intervals.

$$f(x) = y_0 + (x - x_0)\Delta f(x_0) + (x - x_0)(x - x_1)\Delta^2 f(x_0) + (x - x_0)(x - x_1)(x - x_2)\Delta^3 f(x_0) + \dots$$

PART B

1. From the following table find $f(x)$ and hence $f(6)$ using Newton's divided difference

x	1	2	7	8
F(x)	1	5	5	4

2. The population of a town is as follows:

Year x	1941	1951	1961	1971	1981	1991
Population y	20	24	29	36	46	51

Estimate the population increase during the period 1946 to 1976 using

3. Use Newton's forward interpolation formula to find $f(2.75)$ from the following table

x	2.5	3	3.5	4	4.5	5
F(x)	24.145	22.043	20.225	18.644	17.262	16.047

4. Find the value of $\tan 33^\circ$ by using Lagrange's formula of interpolation given :

x	30°	32°	35°	38°
$\tan x$	0.5774	0.6249	0.7002	0.7813

5. From the following table estimate $e^{0.664}$ correct to three decimal using Stirling's formula

x	0.61	0.62	0.63	0.64	0.65	0.66	0.67
e^x	1.840431	1.858928	1.877610	1.896481	1.915541	1.934792	1.954237

6. Find the value of y at $x=1.05$ from the following table

x	1.0	1.1	1.2	1.3	1.4
y	0.841	0.891	0.932	0.964	0.985

UNIT :IV

Numerical Differentiation And Integration.

Part:A

1.What are the errors in trapezoidal rule and Simpsons rule of numerical iteration?

The error in the trapezoidal rule is

$$E < \frac{(b-a)h^2}{12} y''(\xi)$$

The error in the simpsons rule is

$$E < \frac{-h^4}{180}(b-a).$$

2.Compute the value of the definite integral $\int_4^{5.2} \log_e x dx$ or $\int_4^{5.2} \ln x dx$ using Simpsons rule?

Here $x_0 = 4$, $x_0 + nh = 5.2$

$$h=5.2 - 4=0.2$$

x	4	4.2	4.4	4.6	4.8	5	5.2
lnx	1.38629	1.43508	1.48160	1.526056	1.568616	1.609437	1.648658

By simpson's 1/3 rule

$$\int_4^{5.2} \ln x dx = \frac{h}{3} [(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5)]$$

$$= \frac{0.2}{3} [3.034952 + 2(3.050221) + 4(4.570577)]$$

$$=1.827847.$$

3.What is the condition for Simpsons 3/8 rule state the formula?

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{8} [(y_0 + y_n) + 3(y_1 + y_2 + \dots + y_{n-1}) + 2(y_3 + y_6 + \dots + y_{n-3})]$$

4. Write down the expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = x_n$ by Newton's backward difference formula.

$$f'(x_0) = \frac{1}{h} \left[\nabla y_0 + \frac{1}{2} \nabla^2 y_0 + \frac{1}{3} \nabla^3 y_0 + \dots \right]$$

$$f''(x_0) = \frac{1}{h^2} \left[\nabla^2 y_0 + \nabla^3 y_0 + \frac{11}{12} \nabla^4 y_0 + \dots \right]$$

5. Use Simpson's 1/3 rule to estimate the value of $\int_1^5 f(x)dx$ given

x	1	2	3	4	5
f(x)	13	50	70	80	100

$$\int_1^5 f(x)dx = \frac{h}{3} [(y_0 + y_4) + 2(y_2) + 4(y_1 + y_3)]$$

$$= \frac{1}{3} [(13 + 100) + 2(70) + 4(50 + 80)]$$

$$= \frac{1}{3} (113 + 140 + 520)$$

$$= 257.67$$

6. Evaluate $\int_1^4 f(x)dx$ from the following table by Simpson's 3/8 rule

x	1	2	3	4
----------	----------	----------	----------	----------

f(x)	1	8	27	64
-------------	----------	----------	-----------	-----------

$$\int_1^4 f(x)dx = \frac{3h}{8} [(y_0 + y_3) + (y_1 + y_2)]$$

$$= \frac{3}{8} [1 + 3(8) + 3(27) + 64]$$

$$= \frac{3}{8} [1 + 24 + 81 + 64]$$

$$= 63.75$$

7. Write the formula for $\frac{dy}{dx}$ at $x = x_0$ using forward difference operator?

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h} \left[\Delta y_0 + \frac{1}{2} \Delta^2 y_0 + \frac{1}{3} \Delta^3 y_0 + \dots \right]$$

8. State trapezoidal rule to evaluate $\int_{x_0}^{x_n} y(x)dx$.

Trapezoidal rule

$$\int_{x_0}^{x_n} y(x)dx = \frac{1}{2} h [(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})]$$

9. What is the order of error in Trapezoidal formula

Error in the trapezoidal formula is of the order h^2

10. Using trapezoidal rule to evaluate $\int_0^{\pi} \sin x dx$ by dividing the range into 6 equal parts.

$$y = \sin x$$

x	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	π
y	0	0.5	0.8660	1	0.8660	0.5	0

$$\int_0^{\pi} \sin x dx = \frac{h}{6} [(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{\pi}{36} [(0 + 0) + 2(0.5 + 0.8660 + 1 + 0.8660 + 0.5)]$$

$$= 0.65136$$

11. State Simpson's $\frac{1}{3}$ rule.

$$\int_{x_0}^{x_n} y(x) dx = \frac{1}{3} h \{ (y_0 + y_n) + 4(y_1 + y_3 + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots) \}$$

12. In order to evaluate $\int_{x_0}^{x_n} y dx$ by Simpson's 1/3 rule as well as by Simpson's 3/8 rule, what is the restriction on the number of intervals?

Let n= interval

Rule:

Simpson's 1/3 rule: The number of ordinates is odd (or) the interval number is even.

Simpson's 3/8 rule: n is a multiple of 3.

13. Using Simpson's rule find $\int_0^4 e^x dx$ given that $e^0 = 1, e^1 = 2.72, e^2 = 7.39, e^3 = 20.09, e^4 = 54.6$

$$\int_0^4 e^x dx = \frac{h}{3} [(y_0 + y_4) + 2y_2 + 4(y_1 + y_3)]$$

$$= \frac{1}{3} [(154.6 + 1) + 2(7.39) + 4(2.72 + 20.09)]$$

$$= 53.8733$$

14. Why is Trapezoidal rule is so called?

The Trapezoidal rule is so called, because it approximates the integral by the sum of n trapezoids

15. When does Simpson's Rule give a exact result?

Simpson's Rule will give exact result if the entire curve $y = f(x)$ is itself a parabola.

16. State the Trapezoidal rule to evaluate $\int_{x_0}^{x_n} f(x) dx$?

$$\int_{x_0}^{x_n} f(x) dx = \frac{h}{2} [(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})]$$

17. Compare Simpson's one third rule with Trapezoidal method.

<u>Simpson's Rule</u>	<u>Trapezoidal Rule</u>
1. Must accurate.	1. Least accurate
2. Interval of integration must be divided into even number of subintervals.	2. can be divided into any number of sub intervals.

PART II

1. From the following table of values of x and y , find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for $x = 1.05$ and $x = 1.20$ from the following table

x	1.00	1.05	1.10	1.15	1.20	1.25
y	1.00000	1.02470	1.04881	1.07238	1.00544	1.14017

2. Find $f'(x), f''(x)$, at $x=4$ from the following

x	1.5	2.0	2.5	3.0	3.5	4.0
$f(x)$	3.375	7.0	13.625	24.0	38.875	59.0

3. Evaluate $\int_0^1 e^{-x^2} dx$ by dividing the range into 4 equal parts using Simpson's rule?

4. Given the following data, find $y'(6)$.

X	0	2	3	4	7	9
Y	4	26	58	112	466	922

5. Evaluate $\int_0^1 \frac{dx}{1+x^2}$, using Trapezoidal rule with $h = 0.2$. Hence obtain an approximate value of π

Unit : 5

Initial Value Problems for Ordinary Differential Equations

Part:A

1.Using Taylor series method find y at $x = 0.1$ correct to four decimal places from

$$\frac{dy}{dx} = x^2 - y, y(0) = 1 \text{ with } h = 0.1 \text{ compute terms up to } x^4.$$

$$\text{Given : } y' = x^2 - y, x_0 = 0, y_0 = 1, x = 0.1, h = 0.1$$

$$y' = x^2 - y \quad , \quad y'_0 = x_0^2 - y_0 = 0 - 1 = -1$$

$$y'' = 2x - y' \quad , \quad y''_0 = 2x_0 - y'_0 = 0 - (-1) = 1$$

$$y''' = 2 - y'' \quad , \quad y'''_0 = 2 - y''_0 = 2 - (1) = 1$$

$$y^{iv} = -y''', \quad y^{iv}_0 = -y'''_0 = -(1) = -1.$$

$$\begin{aligned} y_1 &= y_0 + hy'_0 + \frac{h^2}{2!} y''_0 + \frac{h^3}{3!} y'''_0 + \frac{h^4}{4!} y^{iv}_0 + \dots \\ y(0.1) &= 1 + 0.1(-1) + \frac{(0.1)^2}{2!}(1) + \frac{(0.1)^3}{3!}(1) + \frac{(0.1)^4}{4!}(-1) + \dots \\ &= 1 - 0.1 + 0.005 + 0.000167 - 0.000004 \\ &= 0.905163 \end{aligned}$$

2. Write the Runge Kutta formula of fourth order to solve $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$?

The Runge Kutta formula of fourth order is

$$\text{If } k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_1 = y_0 + \Delta y$$

3.Using R-K method of second order find $y(0.1)$, when $y' = -y, y(0) = 1$.

Here $x_0=0, y_0=1, h=0.1$
 Given $\frac{dy}{dx} = -y = f(x, y)$

Now, $k_1 = hf(x_0, y_0)$
 $= 0.1(-1) = -0.1$
 $k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$
 $= 0.1[-(y_0 + \frac{k_1}{2})]$
 $= 0.1(-1 + \frac{1}{2})$
 $= 0.1(-1/2)$
 $= -\frac{0.1}{2}$
 $= -0.05$
 $\therefore y_1 = y_0 + \Delta y$
 $= 1 - 0.05$
 $= 0.95$

4. State modified Euler's algorithm to solve $y' = f(x, y), y(x_0) = y_0$ at $x = x_0 + h$.

From modified Euler's formula

$$y_{n+1} = y_n + hf \left[x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n) \right]$$

putting $n = 0$

$$y_1 = y(x_0 + h) = y_0 + hf \left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0) \right]$$

5. Using modified Euler's method, find $y(0.1)$ if $\frac{dy}{dx} = x^2 + y^2, y(0) = 1$.

$$x_0 = 0, y_0 = 1$$

$$\frac{dy}{dx} = x^2 + y^2$$

Modified Euler's formula is

$$y_{n+1} = y_n + hf \left[x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n) \right]$$

Putting $n=0$.

$$y_1 = y(x_0 + h) = y_0 + hf \left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0) \right]$$

$$= 1 + 0.1f \left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2} \right)$$

$$= 1 + 0.1f(0.05, 1.05)$$

$$= 1 + 0.1(1.105)$$

$$= 1.1105$$

6. By Taylor's series method, find $y(0.1)$ given $y' = x + y, y(0) = 1$.

Given $y' = x + y$ and $x_0 = 0, y_0 = 1$
Taylor series formula,

$$y_1 = y_0 + hy'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \frac{h^4}{4!}y^{(4)}_0 + \dots$$

Take $h = 0.1$

$$y' = x + y, \quad y'_0 = x_0 + y_0 = 0 + 1 = 1$$

$$y'' = 1 + y', \quad y''_0 = 1 + y'_0 = 1 + 1 = 2$$

$$y''' = y'', \quad y'''_0 = y''_0 = 2$$

$$y_1 = y(0.1) = 1 + 0.1 + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(2) \\ = 1.1103$$

7. Solve $\frac{dy}{dx} = 1 - y, y(0) = 0$, for $x = 0.1$ by Euler's method?

$$y_{n+1} = y_n + h f(x_n, y_n), \quad h = 0.1$$

$$y_1 = y_0 + 0.1 f(x_0, y_0) \\ = 0 + 0.1 f(0, 0) \\ = 0.1(1) = 0.1$$

8. Write down the third order Taylor's algorithm?

$$y_1 = y_0 + hy'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \dots$$

9. Given $\frac{dy}{dx} = x^2y - 1$ and $y(0) = 1$. Find the value of $y(0.1)$ using Taylor series method?

$$\text{Given } y' = x^2y - 1 \quad x_0 = 0, y_0 = 1$$

Taylor's series formula for y_1 is

$$y_1 = y_0 + hy'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \dots$$

$$\begin{aligned}
 y' &= x^2y - 1, & y_0' &= x_0^2 y_0 - 1 \\
 & & &= 0(1) - 1 = -1 \\
 y'' &= 2xy + x^2y' & y_0'' &= 2x_0 y_0 + x_0^2 y_0' \\
 & & &= 2(0)(1) + (0)(-1) = 0
 \end{aligned}$$

$$\begin{aligned}
 y''' &= 2(xy' + y) + x^2y'' + 2xy' \\
 y_0''' &= 2(x_0' y_0 + y_0) + x_0^2 y_0'' + 2x_0 y_0' \\
 &= 2[0(1) + 1] + 0(0) + 2(0)(-1) \\
 &= 2(1) + 0 = 2 \\
 y_1 &= y(0.1) = 1 + \frac{0.1}{1!}(-1) + \frac{(0.1)^2}{2!}(0) + \frac{(0.1)^3}{3!}(2) \\
 y(0.1) &= 1 - 0.1 + 0 + \frac{0.001}{6}(2) \\
 &= 1 - 0.1 + 0.00033 \\
 y(0.1) &= 0.90033
 \end{aligned}$$

10. Using modified Euler's method compute $y(0.1)$ with $h = 0.1$ from $y' = y - \frac{2x}{y}$, $y(0) = 1$.

$$f(x, y) = y - \frac{2x}{y}$$

$$x_0 = 0, y_0 = 1, x_1 = 0.1, h = 0.1$$

By modified Euler method

$$\begin{aligned}
 y_1 &= y_0 + hf\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right] \\
 &= 1 + (0.1) \left[f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2} \left(y_0 - \frac{2x_0}{y_0}\right)\right) \right] \\
 &= 1 + 0.1 f(0.05, 1 + (0.05)(1 - 0)) \\
 &= 1 + (0.1) [f(0.05, 1.05)] \\
 &= 1 + 0.1 \left[1.05 - 2 \frac{(0.05)}{1.05} \right] \\
 &= 1 + (0.1) [1.05 - 0.0952] \\
 &= 1.09548
 \end{aligned}$$

11. Given $\frac{dy}{dx} + y - x^2 = 0$, $y(0.1) = 0.9052$, $y(0.2) = 0.8213$ find correct to four

decimal places y(0.3) using modified Euler's method?

The modified Euler's formula is

$$y_{n+1} = y_n + hf \left[x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n) \right]$$

Given $x_0 = 0, y_0 = 1$

$$x_1 = 0.1, y_1 = 0.9052$$

$$x_2 = 0.2, y_2 = 0.8213$$

$$f(x, y) = x^2 - y, \quad h = 0.1$$

put $n = 2$ in (1)

$$y_3 = y(0.3) = y_2 + hf \left[x_2 + \frac{h}{2}, y_2 + \frac{h}{2} f(x_2, y_2) \right]$$

$$= 0.8213 + 0.1 f \left[0.2 + \frac{0.1}{2}, 0.8213 + \frac{0.1}{2} f(0.2, 0.8213) \right]$$

$$= 0.8213 + 0.1 f[0.25, 0.8213 + 0.05 f(0.2, 0.8213)] \text{-----(2)}$$

Now, $f(0.2, 0.8213) = (0.2)^2 - 0.8213$

$$= -0.7813$$

$$y_2 + \frac{h}{2} f(x_2, y_2) = 0.8213 + \frac{0.1}{2}(-0.7813) \text{.....(3)}$$

$$= 0.7822$$

Substituting (3) in (2)

$$y_3 = 0.8213 + 0.1 f(0.25, 0.7822)$$

$$= 0.8213 + 0.1(-0.71197)$$

$$y(0.3) = 0.7493$$

12.. By Taylor's series method find y(1.1) given $y' = x + y, y(1) = 0$

Given $y' = x + y$

$$x_0 = 1, y_0 = 0.$$

Taylor's series formula for y_1 is

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \dots \text{-----(1)}$$

Take $h = 0.1$

$$y' = x + y, \quad y_0' = x_0 + y_0 = 1 + 0 = 1$$

$$y'' = 1 + y', \quad y_0'' = 1 + y_0' = 1 + 1 = 2$$

$$y''' = y'', \quad y_0''' = y_0'' = 2$$

$$y_1 = 0 + (0.1)1 + \frac{(0.2)^2}{2}(2) + \frac{(0.1)^3}{6}(2)$$

$$= 0.1103$$

13. Write the merits and demerits of the Taylor method of solution.

Merits :

1. Taylor's formula is easily derived for any order according to our interest
2. The values of $y(x)$ for any x (x need not be at grid points) are easily obtained.

Demerits :

This method suffers from the time consumed in calculating the higher derivatives.

14. Write the formula to find k_2, k_4 in Runge Kutta method of fourth order

$$k_2 = hf \left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2} \right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

PART B

1. Using Runge Kutta method compute $y(0.3)$ GIVEN $\frac{dy}{dx} + y + xy^2 = 0$, $y(0) = 1$ by taking $h = 0.1$?
2. Using Taylor's series method find y at $x=0.1$ given $\frac{dy}{dx} = x^2 - y$, $y(0) = 1$
3. Determine values of y at the pivotel points of the interval $(0,1)$ if y satisfies the boundary value problem $y'''' + 81y = 81x^2$, $y(0) = y(1) = y''(0) = y''(1) = 0$ (take $n = 3$)
4. Using Taylor's series method find y at $x = 0.1$ and 0.2 given $\frac{dy}{dx} = xy + y^2$, $y(0) = 1$ (upto 4 decimal places)
5. Apply fourth order Runge Kutta method to find $y(0.1)$ and $y(0.2)$, given that $\frac{dy}{dx} = x + y$, $y(0) = 1$. Take $h = 0.1$
6. By the method of finite differences ,solve the differential equation $\frac{d^2y}{dx^2} = x - y$ subject to the boundary conditions $y(0) = 0$ and $y(1) = 2$. Compute y for $x = 0.25(0.25)0.75$

7. Using modified Euler's method solve $\frac{dy}{dx} - y^2 = x$, given $y = 1$ at $x = 0$, find y at $x = 0.2$ and 0.4 with $h = 0.2$.

8. Using Euler's method, solve numerically, $y' = x + y$, $y(0) = 1$, for $x = 0.0(0.2)0.6$ check your answer with the exact solution?