MARIA COLLEGE OF ENGINEERING AND TECHNOLOGY, ATTOOR.

MSc SOFTWARE ENGINEERING AND COMPUTER TECHNOLOGY

PARTIAL DIFGFERENTIAL EQUATION AND INTEGRAL TRANSFORMS (XCS 231)

UNIT I

FOURIER SERIES

2 MARK QUESTIONS AND ANSWERS

1.**State the Parseval's identity for the half-range cosine expansion of $\boldsymbol{f(x)}$ in$\boldsymbol{(0,1)}$?**

Solution:

Parseval's identity for the half-range cosine expansion of $f(x)$ in $(0,1)$ is

$$\frac{1}{b-a}\int_a^b [f(x)]^2 dx = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n{}^2$$

2. **Find the value of $\boldsymbol{a_n}$ in the cosine series expansion of $\boldsymbol{f(x) = k}$ in the interval $\boldsymbol{(0,10)}$?**

Solution:

Given $f(x) = k$ in $(0,10)$.Here $l = 10$

$$a_n = \frac{2}{l}\int_0^l f(x) cos\frac{n\pi x}{l} dx$$

$$= \frac{2}{10}\int_0^{10} k cos\frac{n\pi x}{10} dx$$

$$= \frac{k}{5}\left[\frac{sin\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right]_0^{10}$$

$$= \frac{k}{5}\frac{10}{n\pi} sin\frac{10n\pi}{10}$$

$$= \frac{2k}{n\pi} sinn\pi$$

=0.

3. **Determine $\boldsymbol{b_n}$ in the Fourier series expansion of $\boldsymbol{f(x) = \frac{1}{2}(\pi - x)}$ in $\boldsymbol{0 < x < 2\pi}$ with period $\boldsymbol{2\pi}$?**

Solution:

Given $f(x) = \frac{1}{2}(\pi - x)$ in $0 < x < 2\pi$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\sin nx dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)\sin nx dx$$

$$= \frac{1}{2\pi}\left[(\pi - x)\left(\frac{-\cos nx}{n}\right) - \frac{\sin nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[(\pi - 2\pi)\left(\frac{-\cos 2n\pi}{n}\right) - \frac{\sin 2n\pi}{n^2} - \left|(\pi - 0)\left(\frac{-\cos 0}{n}\right) - \frac{\sin 0}{n^2}\right]\right]$$

$$= \frac{1}{2\pi}\left[\frac{\pi}{n} + \frac{\pi}{n}\right]$$

$$= \frac{2\pi}{2n\pi} \quad = \boldsymbol{\frac{1}{n}}$$

4. . Define Root Mean Square value of a function $f(x)$ in $a < x < b$?

Solution:

Root Mean Square value= $\sqrt{\frac{\int_a^b [f(x)]^2 dx}{b-a}}$ in the interval $a < x < b$.

5. **The Fourier series expansion of $f(x)$ in $(0, 2\pi)$ is $f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n}$.Find the Root mean square value of $f(x)$in the interval $(0, 2\pi)$?**

Solution:

Given $f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n}$

$\therefore a_0 = 0 \qquad a_n = 0 \quad b_n = \frac{1}{n}$

Root mean square value of $f(x)$in the interval $(0,2\pi)$ is

$$\bar{y}^2 = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left({a_n}^2 + {b_n}^2\right)$$

$$= 0 + \frac{1}{2}\sum_{n=1}^{\infty}\left(0 + \frac{1}{n^2}\right)$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

6**. State the sufficient condition for a function $f(x)$ to be expressed as Fourier series (or) Explain Dirichlet's conditions?**

Solution:

Any function f(x) can be developed as a Fourier series f(x)= $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cosnx \sum_{n=1}^{\infty} b_n sinnx$ where

(i) $f(x)$ is periodic ,single valued and finite

(ii) $f(x)$ has a finite number of finite discontinuities in any one period and no infinite discontinuity .

(iii) $f(x)$ has at the most a finite number of maxima and minima.

7. Explain periodic function with examples.

A function $f(x)$ is said to have a period p if for all x, $f(x+p) = f(x)$ where p is a positive constant. The least value of $p > 0$ is called the period of $f(x)$

Example:

$$f(x) = sinx$$

$$f(x + 2\pi) = \sin(x + 2\pi) = sinx$$

Here $f(x) = f(x + 2\pi)$

$\therefore$ $sinx$ is a periodic function with period 2π

8**. What is the sum of the Fourier series at a point $x = x_0$ where the function $f(x)$ has a finite discontinuity.**

$$f(x) = \frac{f(x+x_0)+f(x-x_0)}{2}$$

9.Write the formula of Fourier constants for $f(x)$ in $c, c+2l$

$$a_0 = \frac{1}{l} \int_c^{c+2l} f(x)dx$$

$$a_n = \frac{1}{l} \int_c^{c+2l} f(x) \cos\frac{n\pi x}{l} dx$$

$$b_n = \frac{1}{l} \int_c^{c+2l} f(x) sin\frac{n\pi x}{l} dx$$

10.To **what value, the Fourier series corresponding to $f(x) = x^2$ in (0,2π) converges at $x = 0$**.

The Fourier series converges to $\frac{f(0)+f(2\pi)}{2} = 2\pi^2$

11.**Write the formula for finding constants of a Fourier series in (0,2π)**

$$a_0 = \frac{1}{l} \int_{2\pi}^{0} f(x)\, dx$$

$$a_n = \frac{1}{l} \int_{2\pi}^{0} f(x) cosnx\, dx$$

$b_n = \frac{1}{l} \int_{2\pi}^{0} f(x) sinnxdx$

12.**Find the constant a_0 of the Fourier series for the function $f(x) = x$ in 0$\leq x \leq 2\pi$**

$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)dx$

$= \frac{1}{\pi} \int_0^{2\pi} xdx$

$= \frac{1}{\pi} \left[\frac{x^2}{2}\right]_0^{2\pi}$

$= \frac{1}{\pi} \left[\frac{4\pi^2}{2}\right]$

$= 2\pi$

13.**If the Fourier series for the function $f(x) = 0; 0 < \pi$**

$= sinx; 0 < x < 2\pi$

is $f(x) = -\frac{1}{\pi} + \frac{2}{\pi}\left[\frac{cos2x}{1.3} + \frac{cos4x}{3.5} + \cdots \ldots\right|$ .Deduce that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \cdots \infty$.

Putting $x = \frac{\pi}{2}$ we get,

$f\left(\frac{\pi}{2}\right) = -\frac{1}{\pi} + \frac{2}{\pi}\left|-\frac{1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + \cdots\right| + \frac{1}{2}$

$0 = -\frac{1}{\pi} + \frac{2}{\pi}\left|-\frac{1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + \cdots\right| + \frac{1}{2}$

$-\frac{1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + \cdots = \left(\frac{1}{\pi} - \frac{1}{2}\right)\left(\frac{-\pi}{2}\right)$

$= \frac{\pi - 2}{4}$

14.**Find the constant a_0 of the Fourier series for the function $f(x) = k, 0 < x < 2\pi$**

$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)dx$

$= \frac{1}{\pi} \int_0^{2\pi} kdx$

$= \frac{k}{\pi} [x]_0^{2\pi}$

$= \frac{2\pi k}{\pi}$

=2k

15.**If the Fourier series corresponding to $f(x) = x$ in the interval (0,π) is**

$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cosnx + b_n\, sinnx)$ **without finding the values f a_0, a_n, b_n. Find the value of**

$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$

By using Parseval's identity $\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi}\int_0^{2\pi} x^2 dx$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi}$$

$$= \frac{8}{3}\pi^2.$$

16.**For the half-range sine series in the interval $0 < x < l$ the coefficient b_n is given by……**

$b_{n=}\frac{2}{l}\int_0^l f(x)sin\frac{n\pi x}{l}dx$

17. **The functions $f(x) = tanx, f(x) = sin\left(\frac{1}{x}\right)$ cannot be expanded as a Fourier series .Why?**

$tanx$ cannot be expanded as a Fourier series. Since $tanx$ has infinite number of infinite discontinuities. So it does not satisfy one of the Dirichlet's conditions.

18.**Define the value of the Fourier series of $f(x)$ at a point of discontinuity.**

At a point $x = x_0$ where $f(x)$ has finite discontinuity, the sum of the Fourier series=$\frac{1}{2}[y_1 + y_2]$ where $y_1\ and y_2$ are the two values of $f(x)$ at the point of finite discontinuity.

19.If $f(x) = x^2 + x$ is expressed as a Fourier series in the interval $(-2, 2)$ to which value this series converges at $x = 2$

$x = 2$ is a point of discontinuity in the extremum

$$(f(x))_{x=2} = \frac{f(-2)+f(2)}{2}$$

$$= \frac{4-2+4+2}{2}$$

$$=4$$

20.Write parseval's theorem on Fourier constants?

If the Fourier series corresponding to $f(x)$ converges uniformly to $f(x)$ in $(-l, l)$ then

$$\frac{1}{l}\int_{-l}^{l}[f(x)]^2 dx = \frac{{a_0}^2}{2} + \sum_{n=1}^{\infty}\left({a_n}^2 + {b_n}^2\right)$$

UNIT: II

FOURIER TRANSFORM

1.Find the Fourier transform of $f(x) = \begin{cases} e^{ikx}, a < x < b \\ 0, x \le a \; and \; x > b \end{cases}$

Solution:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x).\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{a}^{b} e^{ikx}.\, e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{a}^{b} e^{i(k+s)x} dx \right|$$

$$= \frac{1}{\sqrt{2\pi}} \left[[\frac{e^{i(k+s)x}}{i(k+s)}]_{a}^{b} \right]$$

$$= \frac{1}{\sqrt{2\pi} i(k+s)} [e^{i(k+s)b} - e^{i(k+s)a}]$$

2.Find the Fourier sine transform of $\frac{1}{x}$

Solution:

We know that

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) sinsxdx$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \frac{1}{x} sinsxdx$$

$$= \sqrt{\frac{2}{\pi}} . \frac{\pi}{2}$$

$$= \sqrt{\frac{\pi}{2}}.$$

3.State the Fourier integral theorem.

Solution:

The Fourier integral theorem of f(x) in the interval $(-\ell, \ell)$ is

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\, dx d\lambda$$

4.Write down the Fourier cosine transform pair of formulae.

Solution:

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x) cossxdx$$

$$F^{-1}[F_c[f(x)]] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_c[f(x)] cossxds$$

$F_c[f(x)]$ and $F^{-1}[F_c[f(x)]]$ are called Fourier cosine transform pair.

5.Define Fourier Transform pair.

Solution:

(i) $F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\, dx$

(ii)Inversion formulae

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x)]e^{-isx}\, dx$$

6.Define Fourier Transform sine transform pair formulae.

Solution:

(i) Fourier sine Transform

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x) sinsxdx$$

(ii)Inversion Formulae

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_s[f(x)] sinsxdx.$$

7.State the Fourier transform of the derivatives of a function.

Statement:

The Fourier transform of $F^1(x)$,the derivative of $F(x)$ is $f(s)$, where $f(s)$ is the Fourier transform of $F(x)$.

$F[F^1(x)] = is\, f(s)$.

8.**Give a function which is self reciprocal under Fourier sine and cosine transforms**

$\frac{1}{\sqrt{x}}$

9. State the Parseval's identity on Fourier transform.

Statement:

If $f(x)$ is a given function defined in $(-\infty, \infty)$ then it satisfy the identity

$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$.where F(s) is the Fourier Transform of $f(x)$.

10.State the modulation theorem in Fourier Transform.

Statement:

If F(s) is the Fourier Transform of $f(x)$, then

$F[f(x)\, cosax] = \frac{1}{2}[F(s+a) + F(s-a)]$.

11.What is the Fourier Transform of f(x-a) if the Fourier transform of $f(x)$ is $f(s)$.

Solution:

$F[f(x-a)] = e^{ias} f(s)$.

12. Find the Fourier sine transform of $f(x) = e^{-x}$.

Solution:

We know that $F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) sinsxdx$

$F_s[e^{-x}] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} sinsxdx$

$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{1+s^2}\right]$ $\qquad [\because \int_0^{\infty} e^{-ax} sinbxdx = \frac{b}{a^2+b^2}]$

13.Prove that F $[f(ax)] = \frac{1}{a} F\left(\frac{s}{a}\right)$, $a > 0$.

Solution:

F[f(ax)]= $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax).e^{isx}\,dx$

Put $ax = y$

$adx = dy$

$dx = \frac{dy}{a}$

when $x = -\infty, y = -\infty \; and \; x = \infty, y = \infty$

$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y).e^{is\frac{y}{x}}\frac{dy}{a}$

$=\frac{1}{a}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y).e^{i\left(\frac{s}{a}\right)y}.dy$

$=\frac{1}{a}F\left(\frac{s}{a}\right)$.

14.If $f(x) = e^{-ax}$, $a < 0$, find Fourier sine transform of $f(x)$.

Solution:

$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x) sinsxdx$

$F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax} sinsxdx$ $\qquad [\square \int_0^{\infty} e^{-ax} sinbxdx = \frac{b}{a^2+b^2}]$

$= \sqrt{\frac{2}{\pi}}\left[\frac{s}{s^2+a^2}\right]$

15.State inverse theorem for complex Fourier transform.

Solution:

$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x)].e^{-isx}\,ds.$

is called the inverse formula for the complex Fourier transform of $F[f(x)]$.

16. Let $F_c(s)$ be the Fourier cosine transform of $f(x)$.Prove that

$F_c[f(x)cosax] = \frac{1}{2}[F_c(s+a) + F_c(s-a)]$.

Solution:

$F_c[f(x)cosax] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)cosax\, cossxdx$

$=\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\{\cos(a+s)\,x+\cos(a-s)\,x\}dx$

$=\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(a+s)\,xdx+\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty \mathrm{f(x)}\cos(a-s)\,xdx$

$=\frac{1}{2}[F_c(s+a)+F_c(s-a)]$

17.$F\,[f(x)] = F(s)$ then prove that F$[e^{iax}f(x)] = F(s+a)$.

Solution:

F$[e^{iax}f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iax}f(x).\,e^{isx}\,ds.$

$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x).\,e^{i(s+a)}\,dx$

F $[e^{iax}f(x)] = F(s+a)$

18.What is the sine transform of $f\,(ax)$ if $\overline{f_s}$(s) is the Fourier sine transform of $f(x)$.

Solution:

We know that

$F_s[f(ax)] = \frac{1}{a}F_s(\frac{s}{a})$

Where $F_s[f(x)] = F_s(s)$

Here it is given that $F_s[f(x)] = \overline{f_s}$(s)

$F_s[f(ax)] = \frac{1}{a}\overline{f_s}(\frac{s}{a})$.

19.State Parseval's identity for the half-range cosine expansion of $f(x)$ in $(0,1)$.

Solution:

$2\int_0^1[f(x)]^2\,dx = \frac{a_0^2}{2}+\sum_{n=1}^{\infty} a_n^2$

$a_0 = 2\int_0^1 f(x)dx$

$a_n = 2\int_0^1 f(x)cosnxdx.$

20.Prove that $F_c[f\ (ax)] = \frac{1}{a}F_c\left(\frac{s}{a}\right), a > 0.$

Solution:

$$F[f(ax)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(ax).\,cossx\,dx$$

$ax = y$ when $x = 0, y = 0$

$dx = \frac{dy}{a}$ when $x = \infty, y = \infty$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(y)cos\left(\frac{sy}{a}\right).\frac{dy}{a}$$

$$= \frac{1}{a}\sqrt{\frac{2}{\pi}}\int_0^\infty f(x).\,cos\left(\frac{s}{a}\right)x\,dx$$

$$= \frac{1}{a}F_c\left(\frac{s}{a}\right).$$

UNIT:III

PARTIAL DIFFERENTIAL EQUATIONS

1.Form the partial differential equation by eliminating the arbitrary function from $z = f(xy)$?

Given $z = f(xy)$

Differentiate partially with respect to x: $p = f'(xy)y \ldots\ldots (1)$

Differentiate partially with respect to y: $q = f'(xy)x \ldots\ldots (2)$

Eliminating f from (1)& (2)

(1) □ $\frac{p}{y} = f'(xy)$

(2) □ $\frac{q}{x} = f'(xy)$

$\frac{p}{y} = \frac{q}{x}$

$px - qy = 0$

This is the required partial differential equation.

2.Write down the complete solution of $z = px + qy + c\sqrt{1 + p^2 + q^2}$

Given $z = px + qy + c\sqrt{1 + p^2 + q^2}$

The complete solution is $z = ax + by + c\sqrt{1 + a^2 + b^2}$

3.Obtain partial differential equation by eliminate the arbitrary function from $z = f(x^2 + y^2)$

Given $z = f(x^2 + y^2)$

Differentiate with respect to x: $p = f'(x^2 + y^2)2x$ … … (1)

Differentiate with respect to y: $q = f'(x^2 + y^2)2y$ … … (2)

Eliminating f from (1)& (2)

(1) □ $\frac{p}{2x} = f'(x^2 + y^2)$

(2) □ $\frac{q}{2y} = f'(x^2 + y^2)$

□ $\frac{p}{2x} = \frac{q}{2y}$ □ $\frac{p}{x} = \frac{q}{y}$

$py - qx = 0$

This is the required partial differential equation.

4.Find the complete integral of $p - y^2 = q + x^2$?

Given $p - y^2 = q + x^2$

$p - x^2 = q + y^2 = k$

$\Rightarrow p - x^2 = k; \quad q + y^2 = k$

$\Rightarrow p = x^2 + k; \quad q = -y^2 + k$

By total derivative formula $dz = pdx + qdy$

$\Rightarrow dz = (x^2 + k)dx + (-y^2 + k)dy$

Integrating

$z = kx + \frac{x^3}{3} + ky - \frac{y^3}{3} + C$

5.Form the partial differential equation of all spheres whose centers lie on the z −axis?

The equation of the sphere whose centers lie on the z −axis is

$x^2 + x^2 + (z - c)^2 = r^2$ where r is the constant .

Differentiate partially with respect to x

$2x + 2(z - c)p = 0$ … … (1)

Differentiate partially with respect to x

$2y + 2(z - c)q = 0$ … … (2)

From (2) $z - c = -\frac{y}{q}$

Substituting in (1) we get

$$x - \frac{y}{q}p = 0$$

$$qx = py$$

Which is the required partial differential equation.

6.Form the p.d.e by eliminating the arbitrary constants from $z = ax + by + ab$

Given:$z = ax + by + ab \ldots\ldots\ldots\ldots..(1)$

$$P = \frac{\partial z}{\partial x} = a \ldots\ldots\ldots\ldots\ldots..(2)$$

$$q = \frac{\partial z}{\partial y} = b \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

substituting (2)&(3) in (1) weget the required p.d.e

$$z = px + qy + pq$$

7.Eliminate the arbitrary constants $a\&b$ from $z = ax + by + a^2 + b^2$.

Given: $z = ax + by + a^2+b^2$…………..(1)

Differentiating (1) partially w.r.to'x' we get

$$\frac{\partial z}{\partial x} = a$$

$$P = a \ldots\ldots\ldots\ldots\ldots(2)$$

Differentiating(1) partially w.r.to 'y' we get

$$\frac{\partial z}{\partial y} = b$$

$$q = b \ldots\ldots\ldots\ldots\ldots.(3)$$

Substitute in equation (1) we get the required p.d.e

$$Z = px + qy + p^2 + q^2$$

8. Form a p.d.e by eliminating the arbitrary constants a&b from z=$(x + a)^2 - (y - b)^2$

Given z=$(x + a)^2 - (y - b)^2 \ldots\ldots\ldots\ldots.(1)$

Differentiating (1) partially w.r.to'x' we get

$$\frac{\partial z}{\partial x} = 2(x + a)$$

$P = 2(x + a)$ ………………(2)

Differentiating(1) partially w.r.to 'y' we get

$$\frac{\partial z}{\partial y} = 2(y - b)$$

$q = 2(y - b)$ … … … … … . (3)

Substitute in equation (1) we get the required p.d.e

z=$\left(\frac{p}{2}\right)^2 - (\frac{q}{2})^2$

$\Rightarrow 4z = p^2 - q^2$

9.Find the partial differential equation of all planes having equal intercepts on the x and y axis?

The equation of such plane is $\frac{x}{a} + \frac{y}{a} + \frac{z}{b} = 1$

Partially differentiate with respect to x and y ,we get

$\frac{1}{a} + \frac{p}{b} = 0$

$p = -\frac{b}{a}$ … (1)

$\frac{1}{a} + \frac{q}{b} = 0$

$q = -\frac{b}{a}$ … (2)

From (1) and (2) we get

$p = q$

10.Form a partial differential equation by eliminating the arbitrary constants a and b from the equation $(x - a)^2 - (y - b)^2 = z^2 cot^2 \alpha$

Given $(x - a)^2 - (y - b)^2 = z^2 cot^2 \alpha$ … (1)

Partially differentiate with respect to x and y ,we get

$2(x - a) = 2zpcot^2\alpha$ □ $(x - a) = zpcot^2\alpha \ldots (2)$

$2(y - a) = 2zqcot^2\alpha$ □ $(y - b) = zqcot^2\alpha \ldots (3)$

Substituting (2) and (3) in (1) we get

$(zpcot^2\alpha)^2 + (zqcot^2\alpha)^2 = z^2cot^2\alpha$

$p^2cot^2\alpha + q^2cot^2\alpha = 1$

$p^2 + q^2 = tan^2\alpha$

11.Find the particular integral of $(D^2 - 2DD' + D'^2)z = e^{x-y}$?

Given $(D^2 - 2DD' + D'^2)z = e^{x-y}$

P.I= $\frac{1}{D^2-2DD'+D'^2}e^{x-y}$

$= \frac{1}{1+2+1}e^{x-y}$ $[Replace\ D\ by\ 1\ and\ D'\ by - 1]$

$= \frac{1}{4}e^{x-y}$

12.Solve the partial differential equation $pq = x$

Given $pq = x$

$\frac{p}{x} = \frac{1}{q} = k$

$p = kx$, $q = \frac{1}{k}$

$z = \int pdx + qdy$

$= \int kxdx + \int \frac{1}{k}dy$

$= k\left(\frac{x^2}{2}\right) + \frac{1}{k}y + c$

13. Find the complete integral of $q = 2px$

Given $q = 2px$

$xp = \frac{q}{2} = k$

$p = \frac{k}{x}$ $q = 2k$

$z = \int pdx + qdy$

$= k \int \frac{1}{x} dx + \int 2kdy$

$= klogx + 2ky + c$

14.Form a p. d .e by eliminating the arbitrary constants from z=$ax^2 + ay^2 + b$

Soln: p=$\frac{\partial z}{\partial x}$=$a^2$

q=$\frac{\partial z}{\partial y}$=2ay

(2) ⟹y= $\frac{q}{2a}$

$y^2 = \frac{q^2}{4a^2}$

$y^2 = \frac{q^2}{4a^2}$ by (1)

$4y^2 p = q^2$ which is the required p.d.e.

15.Form the pde by eliminating a and b from $z = a(x + y) + b$

Soln: Given $z = a(x + y) + b$

p=$\frac{\partial z}{\partial x}$=$a$ ………………….(1)

q=$\frac{\partial z}{\partial y}$=a ……………………..(2)

From (1) and (2) we get the required p.d.e. ,

p=q.

16. Form the general solution of $\frac{\partial^2 z}{\partial y^2}$=0

Soln: Given $\frac{\partial^2 z}{\partial y^2}$=0

$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$=0

Integrating p.w.r.to y on both sides

$\frac{\partial z}{\partial y}$=f(x)

Again integrating p.w.r.to y on both sides

z=f(x)y+F(x) where both f(x) and F(x0 are arbitrary.

17.Obtain the complete solution of the equation $z = px + qy - 2\sqrt{pq}$

Soln: Given $z = px + qy - 2\sqrt{pq}$

This is of the form $z = px + qy$+f(p,q)

Hence the complete integral is

$z = ax + by - 2\sqrt{ab}$ where a and b are arbitrary constants.

18. Find the complete integral of pq=xy

Soln: Given pq=xy

Hence $\frac{p}{x} = \frac{y}{q}$

It is of the form f(x,p)= $\varphi(y, q)$

Let $\frac{p}{x} = \frac{y}{q}$ =a

 $p = ax \ \ and \ q = \frac{y}{a}$

$Hence \ \ dz = pdx + qdy$

$dz = axdx + \frac{y}{q}dy$

*Integrating on b*oth sides

z=$a\frac{x^2}{2} + \frac{y^2}{2a} + c$

$2az = a^2x^2 + y^2 + b$ which is the required p.d.e.

19. Solve $(D^2 + 6DD^1 + 9D^2)z = 0$

Soln: The given pde is $(D^2 + 6DD^1 + 9D^2)z = 0$

The A.E is $m^2 + 6m + 9 = 0$

 $(m + 3)(m + 3) = 0$

 $m = -3, -3$

$The\ solution\ is\ z = f_1(y - 3x) + xf_2(y - 3x)$

20.Find the particular integral of $(D^2 + 4DD^1)y = e^x$

Soln; P.I $=\frac{1}{D^2+4DD^1}e^x$

$=\frac{1}{D^2+4DD^1}e^{x+0y}$

$=e^x\left(\frac{1}{1+4(1)(0)}\right)$

$=e^x$

Unit IV

Z-Transform

1.Find z$[(-1)^n]$?

We know that $Z[a^n] = \frac{z}{z-a}$

$$Z[(-1)^n] = \frac{z}{z-(-1)}$$

$$= \frac{z}{z+1}$$

2.Show that Z-Transform is linear?

$$Z[af(n) + bg(n)] = aZ[f(n)] + bZ[g(n)]$$

We know that $Z[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n}$ and $Z[g(n)] = \sum_{n=0}^{\infty} g(n)z^{-n}$

$$Z[af(n) + bg(n)] = \sum_{n=0}^{\infty}[af(n) + bg(n)]z^{-n}$$

$$= \sum_{n=0}^{\infty} af(n)z^{-n} + \sum_{n=0}^{\infty} bg(n)z^{-n}$$

$$= a\sum_{n=0}^{\infty} f(n)z^{-n} + b\sum_{n=0}^{\infty} g(n)z^{-n}$$

$$= aZ[f(n)] + bZ[g(n)]$$

3.Find the Z –Transform of $\left(\frac{-1}{3}\right)^n$?

We know that $Z[a^n] = \frac{z}{z-a}$

$$Z\left[\left(\frac{-1}{3}\right)^n\right] = \frac{z}{z-\left(\frac{-1}{3}\right)}$$

$$= \frac{z}{z+\frac{1}{3}}$$

$$= \frac{3z}{3z+1}$$

4.State final value theorem on Z- Transform?

If $Z[f(n)] = F[z]$ then $\lim_{z\to 1}(z-1)F[z] = \lim_{n\to\infty} f(n)$

5.Find the Z –Transform of $\sin\frac{n\pi}{2}$?

We know that $Z[sinn\theta] = \frac{zsin\theta}{z^2-2zcos\theta+1}$

To find $Z\left[\sin\frac{n\pi}{2}\right]$

$Z\left[\sin\frac{n\pi}{2}\right] = \frac{zsin\frac{\pi}{2}}{z^2-2zcos\frac{\pi}{2}+1}$

$= \frac{z}{z^2+1}$

6.Form the difference equation generated by $y_n = an + b2^n$?

Given $y_n = an + b2^n \dots\dots (1)$

$y_{n+1} = a(n+1) + b2^{n+1} = a(n+1) + b2^n 2 \dots\dots (2)$

$y_{n+2} = a(n+2) + b2^{n+2} = a(n+2) + b2^n 4 \dots\dots (3)$

Eliminating a and b from (1) ,(2) and (3)

$$\begin{vmatrix} y_n & n & 1 \\ y_{n+1} & n+1 & 2 \\ y_{n+2} & n+2 & 4 \end{vmatrix} = 0$$

$y_n[4(n+1) - 2(n+2)] - n[4y_{n+1} - 2y_{n+2}] + 1[\, y_{n+1}(n+2) - y_{n+2}(n+1)] = 0$

$y_n[4n + 4 - 2n - 4] - 4ny_{n+1} + 2n\, y_{n+2} + ny_{n+1} + 2y_{n+1} - ny_{n+2} - y_{n+2} = 0$

$2ny_n - 4ny_{n+1} + 2n\, y_{n+2} + ny_{n+1} + 2y_{n+1} - ny_{n+2} - y_{n+2} = 0$

$(n-1)y_{n+2} + (2-3n)y_{n+1} + 2ny_n = 0$

7.Define Z –Transform of $\{f(n)\}$?

$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

8. Form the difference equation generated by $y_n = a - b3^n$?

Given $y_n = a - b3^n \dots\dots (1)$

$y_{n+1} = a - b3^{n+1} = a - b3^n 3 \dots\dots (2)$

$y_{n+2} = a - b3^{n+2} = a - b3^n 9 \dots\dots (3)$

Eliminating a and b from (1) ,(2) and (3)

$$\begin{vmatrix} y_n & 1 & -1 \\ y_{n+1} & 1 & -3 \\ y_{n+2} & 1 & -9 \end{vmatrix} = 0$$

$y_n[-9+3] - 1[-9y_{n+1} + 3y_{n+2}] - 1[y_{n+1} - y_{n+2}] = 0$

$y_n[-6] + 9y_{n+1} - 3y_{n+2} - y_{n+1} + y_{n+2} = 0$

$-2y_{n+2} + 8y_{n+1} - 6y_n = 0$

$2y_{n+2} - 8y_{n+1} + 6y_n = 0$

9. Find the Z –Transform of $\{n\}$?

We know that $Z[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n}$

$$Z[(n)] = \sum_{n=0}^{\infty}(n)z^{-n}$$

$$= 0 + 1z^{-1} + 2z^{-2} + 3z^{-3} + \cdots$$

$$= \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3}\cdots$$

$$= \frac{1}{z}\left[1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + \cdots\right]$$

$$= \frac{1}{z}\left[1 - \frac{1}{z}\right]^{-2} \qquad [\square\ (1-x)^{-2} = 1 + 2x + 3x^2 + \cdots]$$

$$= \frac{1}{z}\left[\frac{z-1}{z}\right]^{-2}$$

$$= \frac{1}{z}\left[\frac{z}{z-1}\right]^2 = \frac{z^2}{z(z-1)^2} = \frac{z}{(z-1)^2}$$

10.State convolution theorem on Z –Transform?

(i) If $Z[f(n)] = F[z]$ and $Z[g(n)] = G[z]$ then $Z[f(n) \square g(n)] = F[z]G[z]$

(ii) If $Z[f(t)] = F[z]$ and $Z[g(t)] = G[z]$ then $Z[f(t) \square g(t)] = F[z]G[z]$

11. . Find the Z –Transform of $f(n) = \begin{cases} \frac{a^n}{n!} & for\ n \geq 0 \\ 0 & otherwise \end{cases}$

Given $f(n) = \begin{cases} \frac{a^n}{n!} & for\ n \geq 0 \\ 0 & otherwise \end{cases}$

We know that $Z[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n}$

Also we know that $Z[a^n f(n)] = F\left[\frac{z}{a}\right]$

$$Z\left|a^n \frac{1}{n!}\right| = \left\{\sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}\right\}_{z \to \frac{z}{a}}$$

$$= \left\{\frac{1}{0!} + \frac{1}{1!}z^{-1} + \frac{1}{2!}z^{-2} + \frac{1}{3!}z^{-3} + \cdots\right\}_{z \to \frac{z}{a}}$$

$$= \left\}1 + \frac{1}{1!}\left(\frac{1}{z}\right) + \frac{1}{2!}\left(\frac{1}{z^2}\right) + \frac{1}{3!}\left(\frac{1}{z^3}\right) + \cdots\right\}_{z \to \frac{z}{a}}$$

$$= \left\}1 + \frac{1}{1!}\left(\frac{1}{z}\right) + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^2 + \cdots\right\{_{z \to \frac{z}{a}}$$

$$= \left\{e^{\frac{1}{z}}\right\}_{z \to \frac{z}{a}} = e^{\frac{a}{z}}$$

12. State initial value theorem on Z- Transform?

If $Z[f(n)] = F[z]$ then $\lim_{z\to\infty} F[z] = \lim_{n\to 0} f(n)$

13.Form the difference equation generated by $u_n = a2^n$?

Given $u_n = a2^n$ ... (1)

$$u_{n+1} = a2^{n+1} = a2^n 2 = 2u_n \qquad by\ (1)$$

14.Find the value of $Z[f(n)]$ when $f(n) = na^n$?

Given $f(n) = na^n$

We know that $Z[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n}$

Also we know that $Z[a^n f(n)] = F\left[\frac{z}{a}\right]$

$$Z[a^n n] = \{\textstyle\sum_{n=0}^{\infty} nz^{-n}\}_{z \to \frac{z}{a}}$$

$$= \left\{\frac{z}{(z-1)^2}\right\}_{z \to \frac{z}{a}} \qquad \left|\because\ Z[n] = \frac{z}{(z-1)^2}\right|$$

$$= \frac{z/a}{(z/a-1)^2} = \frac{za^2}{a(z-a)^2} = \frac{az}{(z-a)^2}$$

15.Find $Z\left[\frac{1}{n}\right]$?

We know that $Z[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n}$

$$Z\left[\frac{1}{n}\right] = \sum_{n=1}^{\infty} \frac{1}{n}z^{-n}$$

$$= \frac{1}{1}z^{-1} + \frac{1}{2}z^{-2} + \frac{1}{3}z^{-3} + \cdots$$

$$= \left(\frac{1}{z}\right) + \frac{1}{2}\left(\frac{1}{z^2}\right) + \frac{1}{3}\left(\frac{1}{z^3}\right) + \cdots$$

$= \left(\frac{1}{z}\right) + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^2 + \cdots$

$= \frac{\left(\frac{1}{z}\right)}{1} + \frac{1}{2}\frac{\left(\frac{1}{z}\right)^2}{} + \frac{1}{3}\left(\frac{1}{z}\right)^2 + \cdots$

16.Find the Z transform of $(n + 2)$

$Z[n + 2] = Z[n] + Z[2]$

$= \frac{z}{(z-1)^2} + 2Z[1]$

$= \frac{z}{(z-1)^2} + 2\frac{z}{z-1}$

17. State and prove initial value theorem on Z- Transform?

Statement:

If $Z[f(n)] = F[z]$ then $\lim_{z\to\infty} F[z] = \lim_{n\to 0} f(n)$

Proof:

We know that $Z[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n}$

$F[z] = f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots$

$= f(0) + f(1)\frac{1}{z} + f(2)\frac{1}{z^2} + \cdots$

$\lim_{z\to\infty} F[z] = f(0) + 0 + 0 + \cdots$

$= f(0)$

$= \lim_{n\to 0} f(n)$

Hence proved

18.Find the Z transform of $(n + 1)(n + 2)$

$Z[(n + 1)(n + 2)] = Z[n^2 + 2n + n + 2]$

$= Z[n^2 + 3n + 2]$

$= Z[n^2] + 3Z[n] + 2Z[1]$ $\quad$ *by linear property*

$= \frac{Z^2+2}{(Z-1)^3} + \frac{3Z}{(Z-1)^2} + \frac{2Z}{Z-1}$

$$= \frac{Z^2+2+3Z(Z-1)+2Z(Z-1)^2}{(Z-1)^3}$$

$$= \frac{Z^2+2+3Z^2-3Z+2Z(Z^2-2Z+1)}{(Z-1)^3}$$

$$= \frac{2Z^3-7Z+2}{(Z-1)^3}$$

19.Find $Z\left[e^{-iat}\right]$

$$Z\left[e^{-iat}\right] = Z\left[e^{-iat}.1\right]$$

$$= \{Z[1]\}_{Z\to ze^{iaT}}$$

$$= \left\{\frac{Z}{Z-1}\right\}_{Z\to ze^{iaT}}$$

$$= \frac{ze^{iaT}}{ze^{iaT}-1}$$

20.Define convolution?

The convolution of two sequences $f(n)$ and $g(n)$ is defined as
$f(n) * g(n) = \sum_{r=0}^{n} f(r)g(n-r)$

UNIT V

Laplace transform

1.Find the Laplace transform of $\frac{1}{\sqrt{t}}$?

Solution :

We know that $L[t^n] = \frac{\Gamma n+1}{s^{n+1}}$

$$L\left[\frac{1}{\sqrt{t}}\right] = L\left[t^{-\frac{1}{2}}\right]$$

$$=\frac{\Gamma(-\frac{1}{2}+1)}{s^{-\frac{1}{2}+1}} = \frac{\Gamma\frac{1}{2}}{s^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{\sqrt{s}}. \qquad \because \Gamma\frac{1}{2} = \sqrt{\pi}$$

2.State the conditions for the existence of Laplace transform of a function?

Solution:

If a function $f(t)$is continuous and piecewise continuous in a closed interval $a \le t \le b$ and is of exponential order then its Laplace transform exist.

3. Find the Laplace transform of $tsinat$?

Solution:

We know that $L[tf(t)] = -\frac{d}{ds}F(s)$

$$L[tsinat] \quad = -\frac{d}{ds}L[sinat]$$

$$= -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)$$

$$= \frac{2as}{(s^2+a^2)^2}$$

4. Find the Laplace transform of t^2cos3t?

Solution:

We know that $L[t^2f(t)] = \frac{d^2}{ds^2}F[s]$

$$L[t^2cos3t] = \frac{d^2}{ds^2}L[cos3t]$$

$$L[t^2cos3t] = \frac{d^2}{ds^2}\left(\frac{s}{s+9}\right)$$

$$= \frac{2s^3-54s}{(s^2+9)^3}$$

5.Find the Laplace transform of $f(t) = t^2e^{3t}$?

Solution:

Given $f(t) = t^2e^{3t}$

We know that $L[t^2f(t)] = \frac{d^2}{ds^2}F[s]$

$$L[t^2e^{3t}] = \frac{d^2}{ds^2}L[e^{3t}]$$

$$= \frac{d^2}{ds^2}L\left[\frac{1}{s-3}\right]$$

$$= \frac{2}{(s-3)^3}$$

6.State the convolution theorem on Laplace transform?

Solution:

If $f(t)$and $g(t)$are functions defined for $t \geq 0$ then $L[f(t) * g(t)] = L[f(t)].L[g(t)]$

7.State the initial value theorem and final value theorem of Laplace transform?

Solution:

Initial value theorem:

If $L[f(t)] = F(s)$,then $\lim_{t\to 0} f(t) = \lim_{s\to\infty} F(s)$

Final value theorem:

If $L[f(t)] = F(s)$,then $\lim_{t\to\infty} f(t) = \lim_{s\to 0} F(s)$

8.Evaluate $L^{-1}\left[\frac{s}{(s+1)^2+4}\right]$?

Solution:

$L^{-1}\left[\frac{s}{(s+1)^2+4}\right] = e^{-t}L^{-1}\left[\frac{s}{s^2+4}\right]$ $[\because L^{-1}[F(s+a)] = e^{-at}L^{-1}[F(s)]]$

$=e^{-t}cos2t$

9.Find $L^{-1}\left[\frac{1}{s}\left(\frac{1}{s^2+\omega^2}\right)\right]$?

Solution:

We know that Find $L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}[F(s)]dt$

$L^{-1}\left[\frac{1}{s}\left(\frac{1}{s^2+\omega^2}\right)\right] = \int_0^t L^{-1}\left[\frac{1}{s^2+\omega^2}\right]dt$

$=\int_0^t \frac{sin\omega t}{\omega}dt$

$= \frac{-cos\omega t+1}{\omega^2}$

$= \frac{1-cos\omega t}{\omega^2}$

10. Find $L^{-1}[\cot^{-1}(as)]$?

Solution:

We know that $L^{-1}[F(s)] = -\frac{1}{t}L^{-1}[F'(s)]$

$L^{-1}[\cot^{-1}(as)] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}\cot^{-1}(as)\right]$

$L^{-1}[\cot^{-1}(as)] = \frac{-1}{t}L^{-1}\left[\frac{-a}{1+(as)^2}\right]$

$=\frac{a}{t}L^{-1}\left[\frac{1}{1+(as)^2}\right]$

$= \frac{a}{t} \times \frac{1}{a} \sin t/a$ $\{\because L^{-1}[F(as)] = \frac{1}{a} f(\frac{t}{a})\}$

$== \frac{1}{t} \sin t/a$

11. Find $L\left[\sqrt{t} - \frac{1}{\sqrt{t}}\right]$?

Solution:

$L\left[\sqrt{t} - \frac{1}{\sqrt{t}}\right] = L[\sqrt{t}] - L\left[\frac{1}{\sqrt{t}}\right]$ $Linear\ property$

$= \frac{\Gamma n+1}{s^{n+1}} - L\left[t^{\frac{1}{2}}\right]$ $L[t^n] = \frac{\Gamma n+1}{s^{n+1}}$

$= \frac{\Gamma 3/2}{s^{3/2}} - \frac{\Gamma \frac{1}{2}}{s^{\frac{1}{2}}}$

$= \frac{\frac{1}{2}\Gamma 1/2}{s^{3/2}} - \frac{\sqrt{\pi}}{\sqrt{s}}$

$= \frac{\sqrt{\pi}}{2s^{3/2}} - \frac{\sqrt{\pi}}{\sqrt{s}}.$

12. Write the formula for Laplace transform of the periodic function?

Solution:

Let $f(t)$ be a periodic function with period T. Then

$$L[f(t)] = \frac{1}{1-e^{-ST}} \int_0^T e^{-st} f(t) dt$$

13.Find the inverse Laplace transform of $log\left(\frac{s+1}{s-1}\right)$?

Solution:

We know that $L^{-1}[F(s)] = -\frac{1}{t} L^{-1}[F'(s)]$

$L^{-1}\left[log\left(\frac{s+1}{s-1}\right)\right] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds} log\left(\frac{s+1}{s-1}\right)\right]$

$= -\frac{1}{t} L^{-1}\left[\frac{d}{ds}[\log(s+1) - log(s-1)]\right]$

$= -\frac{1}{t} L^{-1}\left[\frac{1}{s+1} - \frac{1}{s-1}\right]$

$$=-\frac{1}{t}\left\{L^{-1}\left[\frac{1}{s+1}\right]-L^{-1}\left[\frac{1}{s-1}\right]\right\}$$

$$=-\frac{1}{t}[e^{-t}-e^{t}]$$

$$=\frac{2}{t}sinht.$$

14. Find the Laplace transform of $\frac{1}{(s+2)^4}$?

Solution:

We know that $[L^{-1}[F(s+a)]=e^{-at}L^{-1}[F(s)]]$

$$L^{-1}\left[\frac{1}{(s+2)^4}\right]=e^{-2t}L^{-1}\left[\frac{1}{s^4}\right]$$

$$=e^{-2t}\frac{t^3}{3!}=e^{-2t}\frac{t^3}{6}.$$

15. Find the Laplace transform of $t^2 2^t$?

Solution:

We know that $L[t^2 f(t)]=\frac{d^2}{ds^2}L[f(t)]$

$$L[t^2 2^t]=\frac{d^2}{ds^2}L[2^t]$$

$$=\frac{d^2}{ds^2}L[e^{tlog2}]$$

$$=\frac{d^2}{ds^2}\frac{1}{s-log2}$$

$$=\frac{2}{(s-log2)^3}.$$

16. Find $L^{-1}\left[\frac{1}{s^2+6s+13}\right]$?

Solution:

$$L^{-1}\left[\frac{1}{s^2+6s+13}\right]=L^{-1}\left[\frac{1}{(s+3)^2+4}\right]$$

$$=e^{-3t}L^{-1}\left[\frac{1}{s^2+4}\right] \qquad [\because L^{-1}[F(s+a)]=e^{-at}L^{-1}[F(s)]]$$

$$=e^{-3t}\frac{sin2t}{2}.$$

17. Find the Laplace transform of $\frac{1-cost}{t}$?

Solution:

If $t \to 0$ $\frac{1-cost}{t} \to 0$

∴Laplace transform exist.

$L\left[\frac{1-cost}{t}\right] = \int_s^\infty L[1-cost]\,ds$ $L\left[\frac{f(t)}{t}\right] = \int_s^\infty L[f(t)]\,ds$

$=\int_s^\infty \{L[1] - L[cost]\}\,ds$

$=\int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+1}\right] ds$

$=\left|logs - \frac{1}{2}\log(s^2+1)\right]_s^\infty$

$=\left|log\frac{s}{(s^2+1)^{\frac{1}{2}}}\right]$

18. Find the inverse Laplace transform of $\cot^{-1}\left(k/s\right)$?

Solution:

We know that $L^{-1}[F(s)] = -\frac{1}{t}L^{-1}\left[F'(s)\right|$

$L^{-1}\left|\cot^{-1}\left(k/s\right)\right] = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\cot^{-1}\left(k/s\right)\right]$

$=-\frac{1}{t}L^{-1}\left|\frac{-1}{1+\left(\frac{k}{s}\right)^2} \times \frac{-k}{s^2}\right|$

$=-\frac{1}{t}L^{-1}\left[\frac{k}{s^2+k^2}\right]$

$=-\frac{1}{t}k.\frac{sinkt}{k}$

$=-\frac{sinkt}{t}$

19.If $L[f(t)] = F(s)$,then prove that $L[f(at)] = \frac{1}{a}F(s/a)$?

Solution:

We know that $L[f(t)] = \int_0^\infty e^{-st}f(t)dt$

$\therefore L[f(at)] = \int_0^\infty e^{-st} f(at)dt$

Put $x = at$

$dx = adt$

$t = 0 \rightarrow x = 0$

$t = \infty \rightarrow x = \infty$

$\therefore L[f(at)] = \int_0^\infty e^{-s\frac{x}{a}} f(x)\frac{dx}{a}$

$== \frac{1}{a}\int_0^\infty e^{-\frac{s}{a}x} f(x)dx$

$=\frac{1}{a} F[{}^{s}/{}_{a}]$.

20.Find $L\left[\int_0^t e^{-t}dt\right]$?

Solution:

We know that $L\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s}$

$L\left[\int_0^t e^{-t}dt\right] = \frac{L[e^{-t}]}{s}$

$= \frac{1}{s(s+1)}$.

PART B (8 MARK QUESTIONS)

1. Use convolution theorem find the inverse Laplace transform of $\frac{s}{(s^2+a^2)^2}$?

2. Find the Laplace transform of $\int_0^\infty \frac{e^{-t} sint}{t} dt$?

3. Using Laplace transform solve $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 6t^2e^{-3t}$ with $y = \frac{dy}{dt} = 0$ when $t = 0$?

4. Find the Laplace transform of the periodic function $f(t) = \begin{cases} t, & 0 \le t \le a \\ 2a - t, & a \le t \le 2a \end{cases}$

5. (i)Find$L^{-1}\left(\frac{s+3}{s^2-4s+13}\right)$?

(ii) Verify the initial value theorem and final value theorem for $f(t) = e^{-t}(t+2)^2$?

6. Find the Laplace transform of $f(t) = \frac{1-e^{-t}}{t}$?

7. Find $L\left|e^{-4t}\int_0^t tsin3tdt\right|$?

8. Solve the integral equation $y(t) = 5sint - 2\int_0^t y(u)\cos(t-u)\,du$?